

Methods of Geometry

Solutions manual

Methods of Geometry

Solutions manual

James T. Smith

San Francisco State University

© 1999 by James T. Smith

Preface

This manual consists of the text from the exercise sections of *Methods of geometry*, with solutions inserted for nearly all exercises. It's intended for use by instructors. It should help you in several ways:

- You can skim it to get an idea of the level of skill and persistence required to complete the exercises in *Methods of geometry*.
- You can read it more closely to determine which exercises to assign.
- It's a source of hints that you can provide when your students need help with an exercise.
- It relieves you of the tedious calculations required for some of the analytic geometry exercises.

Methods of geometry includes few routine exercises. Even those that demonstrate only the simplest concepts often require some creative input, or organizational skills to show that a procedure applies to all cases of a problem. Many professional mathematicians who might find themselves teaching an advanced geometry course have not experienced the preparation in geometry that I enjoyed. The solutions manual can help you find a method that works on an exercise that might seem unassailable at first. I was in that situation once. Fortunately, I had the time then to explore the field and find the lore surrounding these examples, or in some cases, reinvent it. You can get hints from the solutions, consider which ones might benefit individual students, and pass them on.

Frequently I assign an exercise, and point out immediately that I expect students to consider it seriously, become perplexed, and visit me for consultation. When they can discuss a problem, I can suggest an approach. I evaluate proposed solutions critically and point out errors and missing parts, and suggest improvements. This process iterates until the solution is correct or the course is over. Students get the satisfaction of doing *something* entirely right, and I get the satisfaction of seeing that happen.

The solutions in this manual are not the only good ones! Each time I offer the course, I receive for some exercise a solution I've never thought of. When that happens to you, you'll remember it for a long time. So offer hints as suggestions, but encourage and seriously consider students' alternative approaches. You learn from them!

Except for a few solutions whose authors I have acknowledged, all these are my own work. Any errors that you find are my responsibility. I would appreciate your telling me about them.

Please use this manual only for class preparation. The solutions are copyrighted, and their further dissemination is not authorized.

James T. Smith

Contents

Preface	vii
4 Solutions to exercises on elementary geometry	1
4.1 Exercises on the incidence and ruler axioms	2
4.2 Exercises related to Pasch's axiom	6
4.3 Exercises on congruence and perpendicularity	10
4.4 Exercises involving the parallel axiom	14
4.5 Exercises on similarity and Pythagoras' theorem	17
4.6 Exercises on circles and spheres, part 1	22
4.7 Exercises on area	28
4.8 Exercises on volume	36
4.9 Exercises on circles and spheres, part 2	40
4.10 Exercises on coordinate geometry	46
5.11 Solutions to exercises on triangle and circle geometry	55
6.11 Solutions to exercises on plane isometries and similarities	97
7.7 Solutions to exercises on three-dimensional isometries and similarities	123
8.5 Solutions to exercises on symmetry	145

Solutions to exercises on elementary geometry

Chapter 3 was a fast, perhaps breathtaking, recapitulation of elementary geometry. Most of that material you'd already studied. It had become central to your later pursuit of the field, nevertheless somewhat foggy. A few areas were perhaps new to you. The *learn by doing* precept applies to review as well as initial study. To master elementary geometry, you need to draw the figures and work out the proofs in chapter 3. And you need to do *exercises*, to see how these methods apply in practice. The standard type of routine elementary geometry exercise, fit for beginning students new to the material, isn't appropriate for this book. For those, you can consult a good school text,¹ or a workbook devoted to that kind of exercise.² Even better, devise routine exercises yourself. This chapter is a collection of meatier, *nonroutine* problems that complement chapter 3. Most of them require a combination of methods for solution. They're really problems in professional mathematics, not just elementary geometry.

The exercises are collected in this chapter for two reasons. First, some require considerable discussion just to state, and others are clearly peripheral to the main course of the subject. Including them in chapter 3 would distract you from its highly organized flow. Second, the most beneficial problems exercise various combinations of solution methods, and it's sometimes unclear just where in chapter 3 they should be included. Although this chapter is

¹ For example, Moise and Downs 1964.

² For example, Rich 1989.

organized roughly the same way as chapter 3, its latter sections have been shuffled to accommodate this particular problem selection.

Some exercises in this chapter complete discussions begun earlier. For example,

- Exercise 4.5.6 corrects the sophism discussed in section 2.2;
- Exercise 4.6.2 completes the section 3.5 coverage of SSA triangle congruence; and
- Exercise 4.8.3 derives the formula for the volume of a sphere mentioned in section 3.14.

Others introduce entirely new subjects: for example,

- Independence proofs for various axioms in section 4.1;
- Arithmetic, geometric, and harmonic means in 4.5 and 4.6; and
- Spherical trigonometry in section 4.9.

There are a few original exercises here. But for most, the author acknowledges debt to his predecessors. Only a few exercises, with clear heritage, are given footnote citations. Others are old gems, whose origin would be hard to establish. The author has used much of this exercise set for decades. Each year new, different, solutions appear. You're encouraged to dive in, try out ideas old and new, and make these problems *your* old friends.

Hints are provided for a few exercises. Sometimes they suggest an appropriate method. Don't feel constrained to follow that, however, since many problems yield to attack from several directions. Other hints are approximate solutions: for example, "the answer is about 14.14." These help you check the feasibility of your solution. But you need to find the *exact* answer, which isn't obvious, even given the approximation. Indeed, 14.14 is *about* the same as both $11 + \pi$ and $10\sqrt{2}$, but those numbers are different.

4.1 Exercises on the incidence and ruler axioms

Concepts

Models of axioms
Finite geometric systems
Independence of the incidence axioms
Four-dimensional simplices
Ruler placement along a line

The incidence axioms of section 3.1 entail only a small part of Euclidean geometry. It's impossible to single out many applications of that material alone, so the exercises in this section are theoretical.

Axiom I5 states roughly that there are enough points to do *some* three-dimensional geometry. It's worded awkwardly. One reason for that was to facilitate the proof of the more informative lemma 3.1.3 and theorem 3.1.4. Exercise 1 explores alternate wordings. Exercise 2 shows that you can't do without I5, because the remaining axioms can be interpreted to describe a structure—a *model* of those four axioms—that's all in one plane. But according to exercise 3, even *with* I5 you don't get very *much* geometry: I1 to I5 have a model consisting of just four points, so you can't prove from the incidence axioms alone that there are more than four.

Exercise 1. Can you replace axiom I5 by the statement, *There exist four noncoplanar points*? How about, *There exists a plane ε and a point not on ε* ?

Solution. A mathematical system consisting of four points A to D , one line $\{A, B, C, D\}$, and no plane satisfies axioms I1 to I4 and statement

$\mathcal{P}$: *There exist four noncoplanar points,*

but not I5. Thus I1 to I4 and $\mathcal{P}$ don't imply I5. It's easy, however, to derive I5 from I1 to I4 and the statement

$\mathcal{T}$: *There exists a plane ε and a point not on ε ,*

and to derive $\mathcal{T}$ from I1 to I4 and $\mathcal{P}$. Therefore, $\mathcal{T}$ could replace I5. ♦

Exercise 2. Consider the mathematical system depicted by figure 4.1.1. It consists of

- three points O , P , and Q ;
- three lines $\{O, P\}$, $\{P, Q\}$, and $\{Q, O\}$;
- one plane $\{O, P, Q\}$.

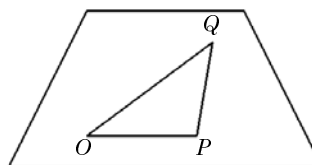
Show that this system satisfies all the incidence axioms except I5.

Solution. It's straightforward. ♦

Exercise 3. Construct a system with four points, six lines, and four planes, that satisfies *all* the incidence axioms.

Solution. Each line contains two points, and each plane, three. ♦

Figure 4.1.1 Finite plane for exercise 2



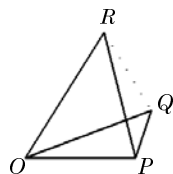


Figure 4.1.2 Finite model
for exercise 4, part 1

Exercises 2 and 4 to 6 together demonstrate that the incidence axioms are *independent*: None is derivable from the others. For each axiom $\mathcal{A}$ there's a model in which $\mathcal{A}$ is false but the rest are true. If you could derive $\mathcal{A}$ from the other axioms, it would also be true in that model.

Exercise 4, Part 1. Consider the system depicted in figure 4.1.2 with four points, five lines, and four planes. Which incidence axioms does it satisfy, and which not? *Careful*: Your answer should depend on whether you interpret the word *three* in I2 as meaning *three distinct*.

Part 2. Modify the system of exercise 3 to produce one with four points, six lines, and three planes, that satisfies all the incidence axioms except I2.

Solution, Part 1. The new system results from the old one for exercise 3 by removing line $\{Q, R\}$. I1 fails because no line contains those points. The systems have the same points and planes, so I4 must hold in the new one. Moreover, any three distinct points that are noncollinear in the new system were noncollinear in the old one. If you regard the word *three* in axiom I2 as meaning *three distinct*, then I2 holds in the new system. If not, then it fails because planes $\{O, Q, R\}$ and $\{P, Q, R\}$ both contain the noncollinear points Q, Q , and R . Axiom I3 holds trivially in the new system because each line contains exactly two points. Clearly, I5 holds in the new system.

Part 2. The new system for this part results from the old one for exercise 3 by removing plane $\{P, Q, R\}$. The systems have the same points and lines, so I1 must hold in the new system. I2 fails because no plane contains the noncollinear points P, Q , and R . I3 and I4 hold because they did in the old system. Clearly, I5 holds in the new system. ♦

Exercise 5. Consider a mathematical system with five points, ten lines each containing just two points, and ten planes each containing just three points. Show that it satisfies all the incidence axioms except I4. Show how to interpret this system as a *simplex*—the simplest polyhedron—in four-dimensional analytic geometry.

Solution. The new system consists of a tetrahedron $OPQR$ like the old system for exercise 3, with each point joined to a new vertex Z in the “fourth dimension.” You can think of its five vertices as the points

$$\begin{aligned}
 O &= \langle 0, 0, 0, 0 \rangle & P &= \langle 0, 1, 0, 0 \rangle & Z &= \langle 1, 0, 0, 0 \rangle \\
 Q &= \langle 0, 0, 1, 0 \rangle \\
 R &= \langle 0, 0, 0, 1 \rangle
 \end{aligned}$$

Planes $\{O, P, Z\}$ and $\{Q, R, Z\}$ have only a single point in common, so I4 fails in the new system. It's easy to verify the remaining axioms. ♦

Exercise 6. Construct a mathematical system that satisfies all the incidence axioms except I3. *Suggestion:* You can do it by adding two points and some lines and planes to the system of exercise 3, and adjusting the content of some of the original lines and planes.

Solution. Start with the old system for exercise 3 and add a point N , as in figure 4.1S.1. Enlarge the line $\{O, P\}$ to $\{N, O, P\}$ and the plane $\{O, P, R\}$ to $\{N, O, P, R\}$ but leave plane $\{O, P, Q\}$ unchanged. Also add lines $\{N, R\}$ and $\{N, Q\}$ and the plane $\{N, Q, R\}$. If you stopped now, $\{N, Q, R\} \cap \{O, P, Q\}$ would be the single point Q , so you must do something more. Add a new point S as in figure 4.1S.2 and enlarge the line $\{N, Q\}$ to $\{N, Q, S\}$ and the planes $\{O, P, Q\}$ and $\{N, Q, R\}$ to $\{O, P, Q, S\}$ and $\{N, Q, R, S\}$. Also add lines $\{O, S\}$, $\{P, S\}$, and $\{R, S\}$ and planes $\{O, S, R\}$ and $\{P, S, R\}$. ♦

Another interesting project along these lines would be to split I1 and I2 each into two axioms (they consist of two sentences each), and explore the independence of the resulting system.

The ruler axiom of section 3.2 excludes finite models like those considered in the previous exercises, since it states that the set of points on any line corresponds to the infinite set of real numbers and shares many properties with it. It says that you can establish such a correspondence—a scale—in at least one way. Exercise 7 shows what other possibilities there are. These relate to the ways you can place a ruler along a line in familiar experience.

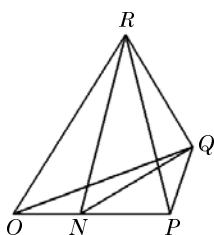


Figure 4.1S.1
Construction for exercise 6

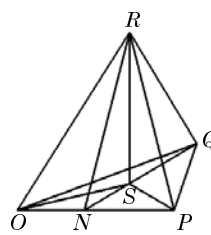


Figure 4.1S.2
Finite model for exercise 6

Exercise 7. Given a scale c for a line g , and real constants a and b with $b = \pm 1$, show that the function d defined by setting $d(P) = a + bc(P)$, for all points P on g , is another scale. Conversely, show that if d is any scale for g , then there are constants a and b such that $b = \pm 1$ and $d(P) = a + bc(P)$ for all P .

Solution. Clearly, if you define d by the formula $d(P) = a + bc(P)$, it will map g one-to-one onto the set of all real numbers. Moreover, for any points P and Q on g , $|d(P) - d(Q)| = |b| |c(P) - c(Q)| = PQ$. Thus d is a scale.

Now suppose d is any scale for g , and let O be the point on g with $d(O) = 0$. For any point P on g ,

$$d(P) = d(P) - d(O) = \pm |d(P) - d(O)| = \pm PO = \pm |c(P) - c(O)|.$$

In fact, the same $+$ or $-$ sign holds in equation $d(P) = \pm [c(P) - c(O)]$ for every point $P \neq O$ on g . For suppose

$$\begin{aligned} d(P) &= c(P) - c(O) \\ d(Q) &= -[c(Q) - c(O)] \end{aligned}$$

for some points P and Q on g . Then

$$\begin{aligned} c(P) - c(Q) &= \pm PQ \\ &= \pm [d(P) - d(Q)] \\ &= \pm [c(P) + c(Q) - 2c(O)]. \end{aligned}$$

Considering the $-$ and $+$ signs in turn, you get $P = O$ or $Q = O$, contrary to hypothesis. Thus, there's a constant $b = \pm 1$ such that $d(P) = -bc(O) + bc(P)$ for all P . ♦

4.2 Exercises related to Pasch's axiom

Concepts

Trihedral angles, their vertices and interiors
 Lines and planes in general position
 Partitioning a plane by lines
 Partitioning space by planes
 Recurrence formulas
 Bounded sets
 Difference equations

An ordered triple $\mathcal{T} = \langle \vec{OP}, \vec{OQ}, \vec{OR} \rangle$ of three noncoplanar rays with a common origin O is called a *trihedral angle*. O is its *vertex*. Clearly, $\mathcal{T}$ determines three angles $\angle POQ$, $\angle QOR$, and $\angle ROP$. (It might seem simpler

to define $\mathcal{T}$ as those angles' union, but that wouldn't let you distinguish one ray of $\mathcal{T}$ from another, which you must do to discuss congruence in section 4.3.)

Exercise 1. Show that a trihedral angle $\mathcal{A}$ determines three dihedral angles, the interiors of any two of which have the same intersection. That intersection is called the *interior* of $\mathcal{A}$.

Solution. It's straightforward. Compare theorem 3.10.1. ♦

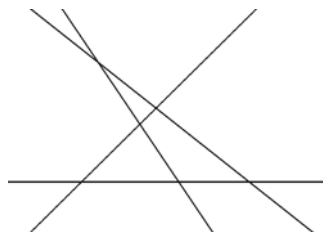
With n distinct points you can divide a line into two rays and $n - 1$ segments that overlap only at common ends. The remaining exercises in this section generalize that familiar result to two and three dimensions. Exercise 6 introduces an algebraic method for solving polynomial difference equations that arise in this study. Give as much detail as possible in your geometric proofs, but don't sacrifice readability and grace. You should be able to confine your methods to those discussed in sections 3.1 to 3.3. These results depend only on the incidence, ruler, and Pasch axioms.

The two-dimensional generalization of the notion of distinct collinear points is that of coplanar lines *in general position*: Each two intersect, but no three pass through the same point. For an example, see figure 4.2.1.

Exercise 2. Show that a finite set $\mathcal{S}$ of n coplanar lines in general position divides their plane ε into a finite number of disjoint partitions. One of these is the union of $\mathcal{S}$, darkened in figure 4.2.1. Each of the others is the intersection of a finite number of sides in ε of members of $\mathcal{S}$. Let p_n be the number of partitions. Show that $p_{n+1} = p_n + n + 1$ for every $n > 0$. Compute p_{10} . *Suggestion:* Starting with no lines and one partition, add lines one by one. Consider the new partitions created when you add the $n+1$ st line. Figure 4.2.1 shows that $p_4 = 12$. You should find $p_{10} = 57$.

Solution. That line g intersects the previous ones at n distinct points, which divide it into $n + 1$ disjoint segments, each of which cuts a partition in half. Thus g adds $n + 1$ new partitions. Start with $p_1 = 3$ and climb to p_{10} . ♦

Figure 4.2.1 Coplanar lines
in general position



A formula such as $p_{n+1} = p_n + n + 1$ that expresses each entry of a sequence $p_1, p_2, \dots$ in terms of one or more of its predecessors is called a *recurrence* formula.

Exercise 3. Express p_n , for $n > 0$, as a polynomial in terms of n . Use this formula to verify your value of p_{10} . Then compute p_{100} . How does the ratio p_{2n}/p_n behave as n increases? *Suggestion:* You should find $p_{100} = 5052$.

Solution. Use the familiar formula $1 + 2 + \dots + n = \frac{1}{2} n(n + 1)$:

$$\begin{aligned} p_1 &= 3 \\ p_2 &= p_1 + 2 = 3 + 2 \\ p_3 &= p_2 + 3 = 3 + 2 + 3 \\ p_4 &= p_3 + 4 = 3 + 2 + 3 + 4 \\ &\vdots \\ p_n &= 3 + 2 + 3 + \dots + n \\ &= 2 + (1 + 2 + 3 + \dots + n) \\ &= 2 + \frac{1}{2} n(n + 1) \\ &= 2 + \frac{1}{2} n + \frac{1}{2} n^2. \end{aligned}$$

p_{2n}/p_n approaches 4 as n increases. ♦

A point set Σ is called *bounded* if there's a number b such that $PQ < b$ for any points P and Q in Σ .

Exercise 4. In general, how many of the partitions in exercise 2 are *unbounded*? How many for $n = 100$? *Suggestion:* For $n = 100$ you should find 201 unbounded partitions.

Solution. For $n = 0$ there's one unbounded partition; for $n = 1$ there are three. In general, the k th line g cuts into two pieces each partition Σ except the union of the lines. If $g \cap \Sigma$ is a segment, then one of the two pieces is bounded, and the other is bounded just in case Σ is bounded. If $g \cap \Sigma$ is a ray, then both pieces of Σ are unbounded. Thus, each new line creates two new unbounded sets, so there are $2n + 1$ unbounded partitions Σ . ♦

A family of planes is said to be *in general position* if the intersection of each three is a point, but no four pass through the same point.

Exercise 5. Show that a finite set $\mathcal{P}$ of planes in general position divides space into a finite number of disjoint partitions. One of these is the union of $\mathcal{P}$. Each of the others is the intersection of a finite number of sides of members of $\mathcal{P}$. Let q_n be the number of partitions. Derive the recurrence

formula $q_{n+1} = q_n + p_n - 1$. Compute q_{10} . *Suggestion:* You should find $q_{10} = 177$.

Solution. Adapt the solution of exercise 2. ♦

An integer constant c and a formula that defines a sequence $d_1, d_2, \dots$ determine a *difference equation* problem: to find a formula for a sequence $f_1, f_2, \dots$ such that $f_1 = c$ and $f_{k+1} - f_k = d_k$ for all k .

Exercise 6. Suppose $d_1, d_2, \dots$ are given by a polynomial of degree n : There are coefficients a_0 to a_n such that $d_k = a_0 + a_1 k + a_2 k^2 + \dots + a_n k^n$ for all k . Show how to find a polynomial solution f of degree $n + 1$ for the difference equation problem $f_1 = c$ and $f_{k+1} - f_k = d_k$ by solving a system of linear equations for the coefficients of f .

Solution. See the exercise 7 solution. ♦

Exercise 7. Use the technique you developed in exercise 6 to find a formula that expresses q_n for $n > 0$, as defined in exercise 5, in terms of n . Use it to verify your value of q_{10} . Then compute q_{100} . How does the ratio q_{2n}/q_n behave as n increases? *Suggestion:* You should find $q_{100} = 166752$.

$$\begin{aligned}
 \text{Solution. } q_n &= an^3 + bn^2 + cn + d \\
 q_{n+1} - q_n &= \frac{1}{2}n^2 + \frac{1}{2}n + 1 \\
 3an^2 + 3an + a + 2bn + b + c &= \frac{1}{2}n^2 + \frac{1}{2}n + 1 \\
 3an^2 + (3a + 2b)n + (a + b + c) &= \frac{1}{2}n^2 + \frac{1}{2}n + 1 \\
 3a &= \frac{1}{2} \quad a = \frac{1}{6} \\
 3a + 2b &= \frac{1}{2} \Rightarrow b = 0 \\
 a + b + c &= 1 \quad c = \frac{5}{6} \\
 d &= 2 \text{ because } q_1 = 3 \\
 q_n &= \frac{1}{6}n^3 + \frac{5}{6}n + 2
 \end{aligned}$$

q_{2n}/q_n approaches 8 as n increases. ♦

Exercise 8. In general, how many of the partitions in exercise 5 are unbounded? How many for $n = 100$? *Suggestion:* For $n = 100$ you should find 9903 unbounded partitions.

Solution. Call the number of bounded sets r_n . Then $r_1 = 3$ and $r_{n+1} = r_n + 2n$ for all n . Applying Exercise 6, you get $r_n = n^2 - n + 3$. ♦

This section's numerical results suggest a guideline for developing algorithms for geometric software. Suppose you've used lines or planes to divide a plane or space into partitions $\Sigma_1, \Sigma_2, \dots$ or $T_1, T_2, \dots$, and you know

that only a few of the partitions have some specific property. If you must perform some operation on just those partitions, then you should try to avoid considering *all* the partitions and testing each in turn for that property. There are so many that such an algorithm could be too inefficient. You should look for some quicker way to identify the appropriate partitions.

Mathematicians have extended and modified the techniques involved in this section to give more detail and to apply to related problems.³

4.3 Exercises on congruence and perpendicularity

Concepts

Angle between a line and a plane
 Congruent trihedral angles
 Congruent tetrahedra

The first two exercises in this section are simple applications of the theory in sections 3.5 and 3.6. Exercises 3 and 4 are a little more involved; those results are used often in more complex arguments about distances in triangles and about perpendicularity in three dimensions. You may find exercise 5 the most challenging in this section. Although stated abstractly, it has a very concrete application in operations research:

Your express company needs to drive four vans each morning from distribution center X to retailers A to D . Assuming that only distances are relevant, where should you locate X to minimize fuel costs?

The remaining exercises lead to a three-dimensional analog of the triangle congruence theory developed in section 3.5. The mathematics isn't difficult, but this study requires attention to organization, and facility with three-dimensional examples. Exercises 7 and 8 are open-ended. The author has never seen a three-dimensional congruence theory published in detail.

Virtually all exercises in the rest of this chapter routinely use the congruence and perpendicularity notions from chapter 3 that are stressed here.

Exercise 1. In figure 4.3.1, X and Y are distinct points in the intersection of distinct planes α and β , A is a point in α but not β , and B is a point in β but not α . Also, $AX = BY$ and $AY = BX$. Prove that $\angle AXB \cong \angle AYB$.

³ See Wetzel 1978 for a summary, and follow its citations for more detail.

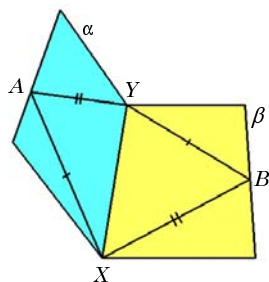


Figure 4.3.1
Exercise 1

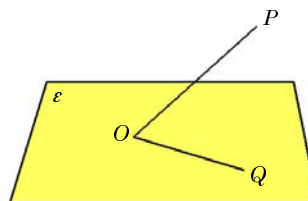


Figure 4.3.2
Exercise 2

Solution. By the SSS congruence theorem, $\Delta ABX \cong \Delta BAY$. ♦

Exercise 2. In figure 4.3.2, O is a point in a plane ε , and P is a point not in ε . Line OP is not perpendicular to ε . Find a point $Q \neq O$ in ε so that $m\angle POQ$ is as small as possible.

Solution. Let Q be the foot of the perpendicular from P to ε , and R be any point on ε but not on ray OQ . Find point S on PR such that $PS = PQ$. Then $PQ < PS$, so $\angle POQ < \angle POS \cong \angle POR$ by the hinge theorem. Q could be any other point on OQ . ♦

The smallest angle in exercise 2 is called the angle *between* $\vec{OP}$ and ε .

Exercise 3, Part 1. Prove that if point X is between but different from vertices A and B of ΔABC , then $CX < AC$ or $CX < BC$.

Part 2. Prove that if point X lies inside ΔABC , then $AX + BX < AC + BC$.

Solution, Part 1. (See figure 4.3S.1.) By the exterior angle theorem, $m\angle AXC > m\angle B$ and $m\angle BXC > m\angle A$. If $AC \leq CX$ and $BC \leq CX$, then by the hinge theorem, $m\angle AXC \leq m\angle A$ and $m\angle BXC \leq m\angle B$. These four angle inequalities entail $m\angle A < m\angle B$ and $m\angle B < m\angle A$ — contradiction!

Part 2. (See figure 4.3S.2.) Locate point B' so that $A-X-B'$ and $XB' = XB$. Then B' lies in the interior of $\angle BXC$, so $m\angle B'XC < m\angle BXC$. Applying the hinge theorem to $\Delta B'XC$ and ΔBXC , you get $B'C < BC$. By the triangle inequality, $AX + BX = AX + B'X = AB' < AC + B'C < AC + BC$. ♦

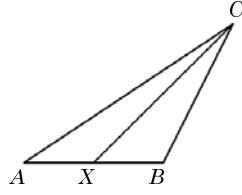


Figure 4.3S.1 Exercise 3
solution, part 1

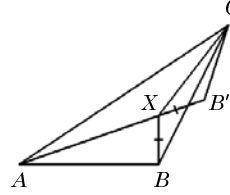


Figure 4.3S.2 Exercise 3
solution, part 2

Exercise 4. Without using the parallel axiom prove that any three of the following conditions on noncoplanar points A to D implies the fourth:

- $\angle B$ of $\triangle ABC$ is right, $\angle C$ of $\triangle ACD$ is right,
 $\angle B$ of $\triangle ABD$ is right, $\angle C$ of $\triangle BCD$ is right.⁴

Solution. Assume the first three conditions; prove $\angle C$ of $\triangle BCD$ is right as follows. Since $\vec{AB} \perp \vec{BC}$ and $\vec{BD}$, theorem 3.5.5 yields $\vec{AB} \perp \vec{BCD}$. By theorem 3.5.8, the perpendicular g to BCD through C is coplanar with $\vec{AB}$, hence g lies in ABC . Since $\vec{CD} \perp g$ and $\vec{AC}$, theorem 3.5.5 yields $\vec{CD} \perp \vec{BC}$.

To prove that $\angle C$ of $\triangle ACD$ is right from the other three conditions, start the same way, to get $\vec{AB} \perp \vec{BCD}$. By theorem 3.5.5, the perpendicular h to ABC through B lies in BCD . By theorem 3.5.8, h is coplanar with the perpendicular k to ABC through C , so k lies in BCD . Now, $\vec{CD}$ and k are perpendiculars to $\vec{BC}$ through C in BCD , hence $\vec{CD} = k$ by theorem 3.5.1, so $\vec{CD} \perp \vec{AC}$. ♦

Exercise 5. Given four coplanar points A to D , find a point X such that $AX + BX + CX + DX$ is as small as possible. *Suggestion:* Treat various configurations separately. For example, consider

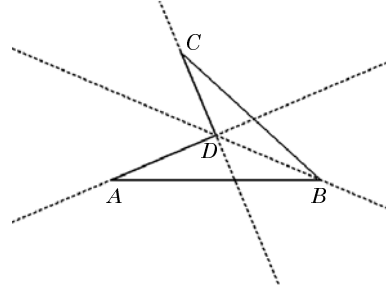
1. all points on a line,
2. D on the edge of $\triangle ABC$ opposite B ,
3. $ABCD$ a convex quadrilateral, and
4. D in the interior of $\triangle ABC$.

Case 4 may prove troublesome. Try subdividing the plane as in figure 4.2.1 and considering X in various partitions.

Solution, Case 1. Name the points so they fall in the order A - B - C - D . You can select any point X such that B - X - C . To see this, make the line the X axis and look at the graph of $AX + BX + CX + DX$.

⁴ This form of exercise 4 is due to H. N. Gupta (1967).

Figure 4.3S.3 Exercise 5
solution, case (4)



Case 2. $AX + BX + CX + DX = (AX + CX) + (BX + DX) \geq AC + BD$, with equality just in case $A-X-C$ and $B-X-D$, i.e. just in case $X = D$.

Case 3 is like case (2), except $X = AC \cap BD$.

Case 4. (See figure 4.3S.3.) Suppose X lies on the same side of $\vec{BD}$ and $\vec{CD}$ as A . Then D lies inside $\triangle BXC$, so

$$\begin{aligned} AX + BX + CX + DX &= (AX + DX) + (BX + CX) \\ &> (AX + DX) + (BD + CD) > AD + BD + CD \end{aligned}$$

by exercise 2 and the triangle inequality. The same result holds if X lies on the same side of $\vec{BD}$ and $\vec{AD}$ as C . Suppose X lies on the same side of $\vec{AD}$ and $\vec{CD}$ as B . Then D lies inside $\triangle AXC$, so

$$\begin{aligned} AX + BX + CX + DX &= (AX + CX) + (BX + DX) \\ &> (AD + CD) + (BX + DX) > AD + CD + BD. \end{aligned}$$

Finally, consider points X on lines $\vec{BD}$, $\vec{CD}$ and $\vec{AD}$. You'll find that

$$AX + BX + CX + DX \geq AD + CD + BD$$

is true for all these X , with equality just in case $X = D$. ♦

Two trihedral angles $\mathcal{T} = \langle \vec{OP}, \vec{OQ}, \vec{OR} \rangle$ and $\mathcal{T}' = \langle \vec{O'P'}, \vec{O'Q'}, \vec{O'R'} \rangle$ are called *congruent* if the angles $\angle POQ, \dots$ of $\mathcal{T}$ are congruent to the corresponding angles $\angle P'O'Q', \dots$ of $\mathcal{T}'$ and corresponding dihedral angles of $\mathcal{T}$ and $\mathcal{T}'$ have the same measure. Trihedral angle congruence is thus conveyed by six equations stating that corresponding angles and dihedral angles have the same measure.

Exercise 6. Prove that if $\langle \vec{OP}, \vec{OQ}, \vec{OR} \rangle$ is a trihedral angle, then $m\angle POQ + m\angle QOR > m\angle ROP$. *Suggestion:* You may want to consider separately cases $m\angle POQ \geq m\angle ROP$ and $m\angle POQ < m\angle ROP$.

Solution. If $m\angle POQ \geq m\angle ROP$, the result is trivial; so assume $m\angle POQ < m\angle ROP$. Find point Q' between R and P so that $\angle POQ' \cong \angle POQ$. If

necessary, relocate Q on $\vec{OQ}$ so that $OQ = OQ'$. Then $\triangle POQ \cong \triangle POQ'$ by the SAS axiom, so $PQ = PQ'$. By the triangle inequality, $PQ + QR > RP$, so $QR > RP - PQ = RP - PQ' = Q'R$. Applying the hinge theorem to $\triangle OQR$ and $\triangle OQ'R$, you get $m\angle QOR > m\angle Q'OR$, hence $m\angle POQ + m\angle QOR > m\angle POQ + m\angle Q'OR = m\angle POQ' + m\angle Q'OR = m\angle ROP$.⁵ ♦

Exercise 7. Find minimal sets of the six trihedral congruence equations that imply the rest. *Suggestion:* Perhaps you can prove that two trihedral angles are congruent if their three pairs of corresponding dihedral angles are. If so, can you get the same result from an even smaller set of equations? Try other possibilities, too.

Solution. The author has never seen this completely worked out. ♦

Two tetrahedra are called *congruent* if their four pairs of corresponding trihedral angles and six pairs of corresponding edges are congruent. Tetrahedral congruence is thus conveyed by six equations stating that corresponding edges have the same length, and sixteen more stating that pairs of corresponding angles have the same measure (twelve pairs of angles determined by vertices, and four pairs of dihedral angles).

Exercise 8. Find minimal sets of the twenty-two tetrahedral congruence equations that imply the rest. Try to organize your results like the triangle congruence theory.

Solution. The author has never seen this completely worked out. Go to it! Good luck! ♦

4.4 Exercises involving the parallel axiom

Concepts

Angle chasing
Properties of parallelepipeds

Except for the last, the exercises in this section aren't *about* parallel lines. But to solve them, you'll need to use results that depend on the parallel axiom. The solution of exercise 1 is an elementary example of the *angle chasing* technique. Exercise 2 looks at first like it might yield to the same method. But it's deceptive and troubling—one of the simpler instances of a genre of problems in which angle chasing seems like tail chasing. The remaining

⁵ This proof is from Legendre [1834] 1848, section VI.XIX.

exercises are three-dimensional. The last, which calls for a three-dimensional analog of the theory of parallelograms developed in section 3.7, is open-ended, although its mathematics is not hard. The author has never seen such a theory worked out in detail.

Exercise 1. Which segment in figure 4.4.1 is shortest? Prove that your choice is correct.

Solution. $\overline{CD}$ is the shortest, by repeated applications of the isosceles triangle, triangle sum, and hinge theorems. ♦

Exercise 2. Suppose E is a point inside square $ABCD$, and $m\angle EAB = 15^\circ = m\angle EBA$, as shown in figure 4.4.2. Without using trigonometry, show that $\triangle CDE$ is equilateral.

Solution. All the square's edges have the same length and its angles all measure 90° . By subtracting angles, $m\angle CBE = 75^\circ = m\angle DAE$. By the isosceles triangle theorem, $AE = BE$. By the SAS Axiom, $\triangle ADE \cong \triangle BCE$, hence $CE = DE$ and $\angle ADE \cong \angle BCE$. Call the measure of these two angles x° . By the triangle sum theorem, $m\angle AED = m\angle BDE = 180^\circ - 75^\circ - x^\circ = 105^\circ - x^\circ$. By angle subtraction, $m\angle CDE = 90^\circ - x^\circ = m\angle DCE$. By the triangle sum theorem, $m\angle CED = 180^\circ - 2(90^\circ - x^\circ) = 2x^\circ$. Apply the hinge theorem twice to get

$$\begin{aligned} x < 30 &\Leftrightarrow 105 - x > 75 &\Leftrightarrow BC > CE \\ &\Leftrightarrow CD > CE &\Leftrightarrow 2x > 90 - x \\ &\Leftrightarrow 3x > 90 &\Leftrightarrow x > 30. \end{aligned}$$

Therefore, the condition $x \neq 30$ is impossible, so $x = 30$ and each angle of $\triangle CDE$ measures 60° . ♦

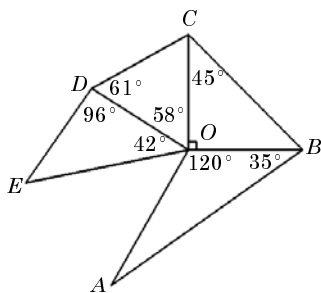


Figure 4.4.1
Exercise 1

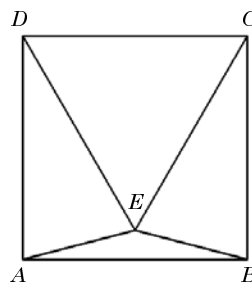


Figure 4.4.2
Exercise 2

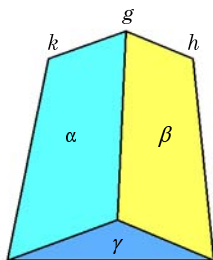


Figure 4.4.3
Exercise 3, part 1

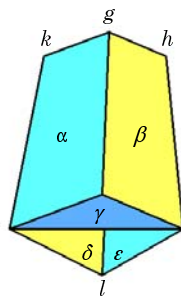


Figure 4.4.4
Exercise 3, part 2

Exercise 3, Part 1. Suppose α , β , and γ are planes intersecting in distinct lines g , h , and k as shown in figure 4.4.3. Using only the incidence axioms, prove that either the three lines are concurrent, or no two of them intersect.

Part 2. Suppose α , β , γ , δ , and ϵ are planes intersecting in distinct lines g , h , k , and l as shown in figure 4.4.4. Prove that g and l are coplanar. How are you stymied when you try to prove this statement using only the incidence axioms?

Solution, Part 1. A point on any two of the intersections lies on all three planes.

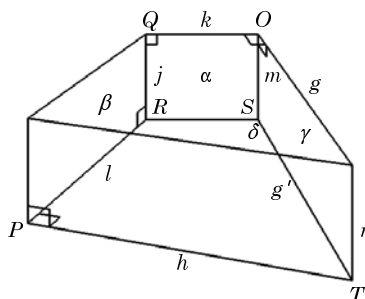
Part 2. If g and h intersect at a point O , then O lies on k and l by part (1), so g and l are coplanar. The same argument works if l and h intersect. If both pairs of lines are parallel, then g and l are parallel, hence coplanar, by the transitivity theorem. ♦

Arthur Winternitz (1940, section II.2) showed by a circuitous argument that the result in exercise 3, part 2, holds even in hyperbolic non-Euclidean geometry—the theory you get by replacing the parallel axiom by its negation. Thus, there ought to be a solution of part 2 that uses only the incidence, ruler, Pasch, and congruence axioms. You're challenged to find one. Winternitz showed that you can't derive it from just the incidence axioms.

Exercise 4. Prove that two noncoplanar lines have exactly one common perpendicular. *Suggestion:* Try building a right prism whose bases are quadrilaterals $\mathcal{Q}$ and $\mathcal{R}$ with two right angles each, such that $\mathcal{Q}$ has an edge on one given line and $\mathcal{R}$ has one on the other.

Solution. Select points O and P on noncoplanar lines g and h as in figure 4.4S.1, and construct planes $\alpha \perp g$ and $\beta \perp h$ through O and P . Then $\alpha \cap \beta$ is a line j , else g and h would be parallel. Through O

Figure 4.4S.1
Exercise 4 solution



and P construct k and $l \perp j$, intersecting j at points Q and R . Then $g \perp k$ and $h \perp l$ and these pairs of lines lie in planes γ and δ . You can prove that $\gamma, \delta \perp \alpha, \beta, j$. Erect $m \perp \gamma$ through O , so that $m \cap \delta$ is a point S . Construct $g' \parallel g$ through S , so that g' is not parallel to h . You can show that g' lies in δ , hence $g' \cap h$ is a point T . Construct $n \perp \delta$ through T . Then show that $n \perp g, h$.

Clearly, lines with two distinct perpendiculars must be coplanar. ♦

Exercise 5. Formulate and prove a set of theorems about parallelepipeds analogous to theorems 3.7.17–3.7.20 about parallelograms. More detail may be appropriate, because there are more concepts in three dimensions:

<i>Two dimensions</i>	<i>Three dimensions</i>
parallelogram	parallelepiped
	right prism with parallelogram bases
	rectangular prism
rectangle	box
	prism with two rhombic bases
	right rhombic prism
rhombus	cuboid (all faces rhombic)
square	cube

Solution. The author has never seen this done in complete detail. ♦

4.5 Exercises on similarity and Pythagoras' theorem

Concepts

Arithmetic, geometric, and harmonic means
Right-triangle trigonometry applications
Correcting the section 2.2 sophism

This section contains exercises that use similarity and Pythagoras' theorem in various ways. In section 4.6 you'll find more exercises that stress Pythagoras' theorem. The techniques emphasized here will be used routinely in the rest of this chapter.

Exercise 1 demonstrates geometrically some properties of three *mean* operations used in many applications. The average $m = \frac{1}{2}(x + y)$ of two numbers x and y is called their *arithmetic* mean because the triple $\langle x, m, y \rangle$ forms an *arithmetic* series: $y - m = m - x$. The *geometric* mean of x and y is the number g such that $\langle x, g, y \rangle$ forms a *geometric* series: $y/g = g/x$, so $g^2 = xy$, hence $g = \sqrt{xy}$. The *harmonic* mean of x and y is the number h whose reciprocal is the average of those of x and y :

$$\frac{1}{h} = \frac{1}{2} \left(\frac{1}{x} + \frac{1}{y} \right).$$

Exercise 1, Part 1. Given x and y with $x < y$ as in figure 4.5.1, show that $h = 2z$ is their harmonic mean. Show geometrically that $x < h < m$. Show algebraically that $\langle h, g, m \rangle$ forms a geometric series, so that $x < h < g < m < y$.

Part 2. Let $ABCD$ be a trapezoid with bases of length $AB = x$ and $CD = y$. Let the line parallel to the bases through the intersection of the diagonals meet the other edges at points E and F . Show that EF is the harmonic mean h .

Solution, Part 1. Label the segments of the common base of the larger right triangles as in figure 4.5S.1. Then

$$1 > \frac{x}{y} = \frac{x/z}{y/z} = \frac{(a+b)/b}{(a+b)/a} = \frac{a}{b},$$

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{z} \left(\frac{z}{x} + \frac{z}{y} \right) = \frac{1}{z} \left(\frac{b}{a+b} + \frac{a}{a+b} \right) = \frac{1}{z}. \quad (1)$$

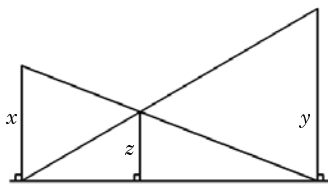


Figure 4.5.1 Constructing the harmonic mean

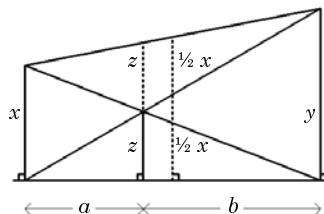


Figure 4.5S.1
Exercise 1 solution

Since $a < b$, segment z stands to the left of the midpoint of the base, $\frac{1}{2}h = z > \frac{1}{2}x$, $h = 2z < \frac{1}{2}(x + y) = m$, and $x < h < m < y$. You can show algebraically that

$$h = \frac{2xy}{x+y} \quad \frac{m}{g} = \frac{g}{h} = \frac{x+y}{2\sqrt{xy}}.$$

Part 2. Find segments of length a and b related to the trapezoid so that equations (1) hold. ♦

Exercise 2, Part 1. Prove that if A to D are distinct noncollinear (not necessarily coplanar) points, and

$$\begin{array}{ll} A' \text{ is the midpoint of } \overline{AB}, & B' \text{ is the midpoint of } \overline{BC}, \\ C' \text{ is the midpoint of } \overline{CD}, & D' \text{ is the midpoint of } \overline{DA}, \end{array}$$

then $A'B'C'D'$ is a parallelogram or A' to D' are collinear and $A'B' = C'D'$.

Part 2. Prove that if A to F are distinct points, no four of which are collinear,

$$\begin{array}{ll} A' \text{ is the midpoint of } \overline{AB}, & D' \text{ is the midpoint of } \overline{DE}, \\ B' \text{ is the midpoint of } \overline{BC}, & E' \text{ is the midpoint of } \overline{EF}, \\ C' \text{ is the midpoint of } \overline{CD}, & F' \text{ is the midpoint of } \overline{FA}, \end{array}$$

and O is a point such that $A'B'C'O$ is a parallelogram, then $D'E'F'O$ is also a parallelogram.

Solution, Part 1. Suppose no three of these points are collinear. By theorem 3.9.2, $g = A'B' \parallel AC$ and $h = C'D' \parallel AC$, so $g = h$ or $g \parallel h$. Similarly, $j = B'C' \parallel BD$ and $k = D'A' \parallel BD$, so $j = k$ or $j \parallel k$. If $g \parallel h$ and $j \parallel k$, then $A'B'C'D'$ is a parallelogram. The other possibilities lead to the alternative result.

Part 2. Show that O is the midpoint of $\overline{AD}$. ♦

Exercise 3. The measures of the angles and edges of a triangle are sometimes called its *parts*. Construct two incongruent triangles such that five parts of one equal five of the other. Describe all possible ways to do this.

Solution. You can't have three pairs of congruent edges, else the SSS theorem would yield congruence. Thus you must have two pairs of congruent edges and three of congruent angles. By the AA similarity theorem, the triangles must be similar. Make one with edges 1, x , and y so that $1 \leq x \leq y$. The other can have corresponding edges x , y , and z . You must have $1/x = x/y = y/z$, so $y = x^2$ and $z = x^3$.

What values are possible for x ? You must have $1 + x > y = x^2$, i.e. $x^2 - x - 1 < 0$. If that's true, then $x + y = x(1 + x) > x^3 = z$, and $y + z =$

$x^2(1+x) > 1+x > 1$, so you can in fact make the triangle. The polynomial $x^2 - x - 1$ has roots

$$\frac{1 \pm \sqrt{5}}{2} \approx \begin{cases} 1.62 \\ -0.62 \end{cases},$$

so you can use any x such that $1 < x < \frac{1}{2}(1 + \sqrt{5})$. ♦

Theorem 3.13.15 provided a summary of right-triangle trigonometry. Exercise 4 demonstrates all cases of its use in practical applications. Exercise 5 shows a typical application: measuring some particularly symmetric figures that you'll study in more detail in chapter 8.

Exercise 4. Consider right triangle $\triangle ABC$ with legs $a = BC$, $b = CA$, hypotenuse $c = AB$, and acute angles $\angle A$ and $\angle B$.

Part 1. Given $a = 10$ and $b = 20$, find c , $m\angle A$, and $m\angle B$.

Part 2. Given $b = 20$ and $c = 30$, find a , $m\angle A$, and $m\angle B$.

Part 3. Given $a = 10$ and $m\angle A = 40^\circ$, find b , c , and $m\angle B$.

Part 4. Given $b = 20$ and $m\angle A = 40^\circ$, find a , c , and $m\angle B$.

Part 5. Given $c = 30$ and $m\angle A = 40^\circ$, find a , b , and $m\angle B$.

Solution, Part 1. $c = \sqrt{a^2 + b^2} = 10\sqrt{5} \approx 22.36$
 $\tan m\angle A = a/b = \frac{1}{2} \Rightarrow m\angle A \approx 26.57^\circ$
 $m\angle B = 90^\circ - m\angle A \approx 63.43^\circ$.

Part 2. $a = \sqrt{c^2 - b^2} = 10\sqrt{5} \approx 22.36$
 $\cos m\angle A = b/c = \frac{2}{3} \Rightarrow m\angle A \approx 48.19^\circ$
 $m\angle B = 90^\circ - m\angle A \approx 41.81^\circ$.

Part 3. $a/b = \tan m\angle A \Rightarrow b = a/\tan m\angle A \approx 11.92$
 $a/c = \sin m\angle A \Rightarrow c = a/\sin m\angle A \approx 15.56$
 $m\angle B = 90^\circ - m\angle A = 40^\circ$.

Part 4. $a/b = \tan m\angle A \Rightarrow a = b \tan m\angle A \approx 16.78$
 $b/c = \cos m\angle A \Rightarrow c = a/\cos m\angle A \approx 26.11$
 $m\angle B = 90^\circ - m\angle A = 40^\circ$.

Part 5. $a/c = \sin m\angle A \Rightarrow a = c \sin m\angle A \approx 19.28$
 $b/c = \cos m\angle A \Rightarrow b = c \cos m\angle A \approx 22.98$
 $m\angle B = 90^\circ - m\angle A = 40^\circ$. ♦

Exercise 5. Find the measures of the dihedral angles formed by the faces of a regular tetrahedron and of a regular octahedron. The latter polyhedron consists of two square pyramids with a common base and eight equilateral faces. *Suggestion:* To gain insight and check the reasonableness of your solutions, you may want to make stiff cardboard models. More precise and robust

models of these and other commonly studied polyhedra are available as kits. You glue together plastic face tiles. To fit precisely, they should be manufactured with the edges beveled to produce the correct dihedral angles. (See the discussion in section 8.5 under the heading

Solution. In regular tetrahedron $ABCD$ with edge 1, find the foot P of the perpendicular from D to plane ABC , and the foot Q of the perpendicular from P to edge $\overline{AB}$. Then $BQ = \frac{1}{2}\sqrt{3}$. The required measure is $\theta = m\angle PQD$:

$$\frac{PQ}{CQ} = \frac{CQ}{BQ} \Rightarrow \frac{PQ}{\frac{1}{2}} = \frac{\frac{1}{2}}{\frac{1}{2}\sqrt{3}} \Rightarrow PQ = \frac{1}{2\sqrt{3}}$$

$$\cos \theta = \frac{PQ}{DQ} = \frac{PQ}{BQ} = \frac{\frac{1}{2\sqrt{3}}}{\frac{1}{2}\sqrt{3}} = \frac{1}{3} \Rightarrow \theta = \arccos \frac{1}{3} \approx 70.52^\circ.$$

In a square pyramid with equilateral triangular faces and edge 1, find the foot P of the perpendicular from the apex D to the base, and the foot Q of the perpendicular from P to a base edge $\overline{AB}$. Then $DQ = \frac{1}{2}\sqrt{3}$ and $PQ = \frac{1}{2}$, so $DP = \frac{1}{2}\sqrt{2}$ by Pythagoras' theorem. The required measure is $\theta = 2m\angle PQD$:

$$\cos m\angle PQD = \frac{PQ}{DQ} = \frac{\frac{1}{2}}{\frac{1}{2}\sqrt{3}} = \frac{1}{\sqrt{3}} \Rightarrow \theta = 2 \arccos \frac{1}{\sqrt{3}} \approx 109.47^\circ. \blacklozenge$$

Exercise 6 is the result you need to correct the sophism in section 2.2. It's included here only because part of the solution—already given in 2.2—uses Pythagoras' theorem. The argument you must supply here uses only methods covered through section 3.7.

Exercise 6. Consider $\triangle ABC$ with $CA < AB$. Prove that the bisector of $\angle A$ meets the perpendicular bisector of $\overline{BC}$ at a point X on the opposite side of $\overline{BC}$ from A . Moreover, the feet Y and Z of the perpendiculars from X to $\overline{CA}$ and $\overline{AB}$ lie outside and inside the triangle legs, respectively.

Solution. By the crossbar theorem, the angle bisector g meets $\overline{BC}$ at a point W , shown in figure 4.5S.2. By the hinge theorem, $m\angle B < m\angle C$. By the exterior angle and triangle sum theorems,

$$m\angle BWA = \frac{1}{2} m\angle A + m\angle C > \frac{1}{2} m\angle A + m\angle B = 180^\circ - m\angle BWA$$

$$2m\angle BWA > 180^\circ.$$

Thus, $\angle BWA$ is obtuse, and g isn't parallel to the perpendicular bisector—they meet at some point X .

spheres. Section 4.9 contains exercises on plane and spherical trigonometry, the value of π , and the curved areas of cylinders, cones, and spheres.

Exercise 1 continues the study of arithmetic, geometric, and harmonic means begun in exercise 4.5.1. It was included in the work of the Alexandrian mathematician Pappus about A.D. 320.⁶

Exercise 1. Given $x < y$, locate points A to C so that $A-B-C$, $AB = x$, and $BC = y$. Let O be the midpoint of $\overline{AC}$, and D be the intersection of the perpendicular to $\overline{AC}$ through B with a semicircle whose diameter is $\overline{AC}$. Let E be the foot of the perpendicular to $\overline{OD}$ through B . Among the segments determined by points A to E and O , find segments of length

- h the harmonic mean of x and y ,
- g their geometric mean, and
- m their arithmetic mean.

Point out geometric meanings of the inequalities $x < h < g < m < y$.

Solution. First, as in figure 4.6S.1, radius OD is half the diameter: $OD = \frac{1}{2}(AB + BC) = \frac{1}{2}(x + y) = m$. Second, $\triangle ABD \sim \triangle DBC$, so $AB/BD = DB/BC$. That is, $x/BD = BD/y$, so $BD = g$. Third, $\triangle BED \sim \triangle OBD$, so $ED/EB = BD/BO$ and $ED = EB(BD/OB) = BD(EB/OB)$. Moreover, $\triangle BOE \sim \triangle DOB$, so $EB/OB = BD/OD$. Thus $ED = (BD)^2/OD = g^2/m = g(g/m) = g(h/g) = h$. Inequality $x < h$ says that $AB < h = ED$, which is equivalent to $OB > EO$: hypotenuse OB of right $\triangle BOE$ is longer than leg EO . Inequality $h < g$ says that leg $h = ED$ of right $\triangle BED$ is shorter than its hypotenuse $g = BD$. Inequality $g < m$ says that leg $g = BD$ of right triangle $\triangle DBO$ is shorter than its hypotenuse $m = OD$. Inequality $m < y$ says that $y = BC$ is longer than the radius $m = OC$. ♦

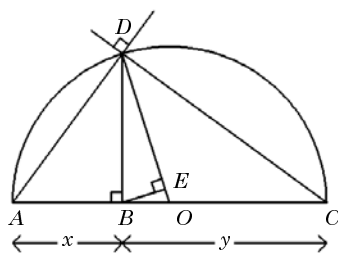


Figure 4.6S.1
Exercise 1 solution

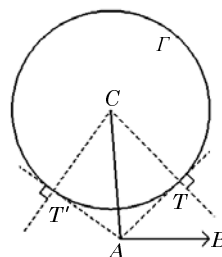


Figure 4.6S.2 Exercise 2
solution, part 1

⁶ See Eves 1963, volume 1, problem 1.5-17.

Although exercise 2 isn't *about* circles, you may want to use some of their properties to solve it. This result completes the triangle congruence theory begun in section 3.5. There you reviewed the SAS, ASA, SSS, and SAA theorems. An example indicated that there's no analogous SSA theorem. Nevertheless, as exercise 2 shows, SSA data can be useful. Sometimes they're enough to determine all the other parts of the triangle. In the other cases, you can *almost* do that; you've only two choices.

Exercise 2. Suppose you have SSA data about $\triangle ABC$: $a = BC$, $b = CA$, and $m\angle A$.

Part 1. Suppose $\angle A$ is obtuse or right. Prove that $a > b$ and that all triangles with this same SSA data are congruent to $\triangle ABC$.

Part 2. Suppose $\angle A$ is acute. Let d be the distance from C to $\vec{AB}$. Show that $d < b$ and $d \leq a$. Prove that if $a = d$ or $a \geq b$, then all triangles with this same SSA data are congruent to $\triangle ABC$. Show that if $d < a < b$, then there's a triangle T incongruent with $\triangle ABC$ such that all triangles with this same SSA data are congruent to $\triangle ABC$ or to T .

Solution, Part 1. $\angle A$ is obtuse or right. Consider the circle Γ in figure 4.6S.2 with center C and radius a . Suppose $a < b$. Then A would lie outside Γ , and you could find points T and T' where the tangents through A would touch Γ . Except for those points, Γ would lie inside $\angle TAT'$. Triangles $\angle CTA$ and $\angle CT'A$ would be right, and $\angle TAC$ and $\angle T'AC$ acute. Since $\angle CAB$ is obtuse, $\vec{AB}$ would lie outside $\angle TAT'$. But B lies on Γ because $BC = a$ —contradiction!

Therefore, $a > b$. Suppose $\triangle A'B'C'$ has the same SSA data: $a = B'C'$, $b = C'A'$, and $m\angle A = m\angle A'$. As in figure 4.6S.3, find the feet D and D' of the perpendiculars to $\vec{AB}$ and $\vec{A'B'}$ through C and C' . Because $\angle A$ and $\angle A'$ are obtuse, D and D' lie outside the triangles: $D-A-B$ and $D'-A'-B'$. Right triangles $\triangle ADC$ and $\triangle A'D'C'$ are congruent because they have congruent hypotenuses and $\angle CAD \cong \angle C'A'D'$. Thus $DC = D'C'$ and $AD = A'D'$. Right triangles $\triangle BDC$ and $\triangle B'D'C'$ are congruent

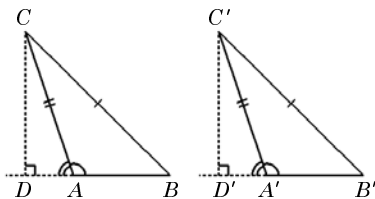


Figure 4.6S.3 Exercise 2
solution, part 1, continued

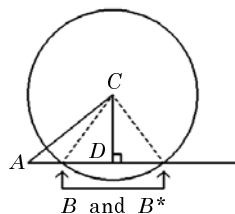


Figure 4.6S.4 Exercise 2
solution, part 2

because they have congruent hypotenuses and a pair of congruent legs, so $DB = D'B'$. It follows that $AB = DB - DA = D'B' - D'A' = A'B'$. By the SSS theorem, $\triangle ABC \cong \triangle A'B'C'$.

Part 2. $\angle A$ is acute. Let D be the foot of the perpendicular to $\vec{AB}$ through C , so that $d = CD <$ the distance from C to any other point on $\vec{AB}$. Thus, $d \leq a$ and $d \leq b$. If $d = b$, then $A = D$ and $\angle A$ would be right—contradiction! Thus $d < b$.

If $a = d$, then $B = D$ and $\angle B$ is right. Let D' be the foot of the perpendicular to $\vec{A'B'}$ through C' . By the SAA theorem, $\triangle ABC \cong \triangle A'D'C'$, so $B'C' = BC = D'C'$, hence $B' = D'$ and $\triangle ABC \cong \triangle A'B'C'$, as claimed.

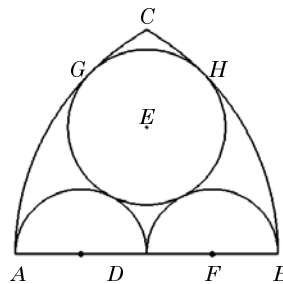
In case $a \geq b$, the hinge theorem implies $m\angle B \leq m\angle A$, so $\angle B$ is also acute, and the foot D of the perpendicular to $\vec{AB}$ through C lies between A and B . You can adapt the argument in the second paragraph of the part 1 solution to show that any triangle with this same SSA data is congruent to $\triangle ABC$.

Finally, consider the case $d < a < b$, as in figure 4.6S.4. D lies inside Γ and A outside, so $\vec{AB}$ intersects Γ at two points, B and another point B^* . One of these lies between A and D , and the other lies on the side of D opposite A . You can adapt the argument in the second paragraph of part 1 to show that any triangle with this same SSA data is congruent to $\triangle ABC$ or $\triangle AB^*C$. ♦

Exercises 3 and 4 seem to relate to church window design and to packing balls in a box. They're really exercises on Pythagoras' theorem.

Exercise 3. In figure 4.6.1, arcs $\widehat{AGC}$ and $\widehat{BHC}$ have centers B and A . The semicircles with centers D and F on $\overline{AB}$ have radius $\frac{1}{4}s$, where $s = AB$. The circle with center E is tangent to the first two arcs and the semicircles. Find its radius r in terms of s .

Figure 4.6.1
Exercise 3



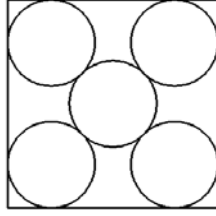


Figure 4.6.2
Quincunx

Solution. Let M be the midpoint of $\overline{AB}$. Then $\triangle AME$ is right, with hypotenuse $s - r$ and legs $\frac{1}{2}s$ and x , hence

$$(s - r)^2 = (\frac{1}{2}s)^2 + x^2. \quad (*)$$

Further, $\triangle DME$ is right, with legs $\frac{1}{4}s$ and x and hypotenuse $\frac{1}{4}s + r$, hence

$$x^2 + (\frac{1}{4}s)^2 = (\frac{1}{4}s + r)^2 = (\frac{1}{4}s)^2 + 2 \cdot \frac{1}{4}s r + r^2,$$

and $x^2 = \frac{1}{2}rs + r^2$. With equation (*) this yields

$$s^2 - 2rs + r^2 = \frac{1}{4}s^2 + \frac{1}{2}rs + r^2,$$

i.e. $\frac{3}{4}s^2 = \frac{5}{2}rs$, hence $r = \frac{3}{10}s$. ♦

Exercise 4, Part 1. In figure 4.6.2, called a *quincunx*,⁷ the square has edge s and the circles, radius r . Find r in terms of s . *Suggestion:* $r \approx 0.207s$.

Part 2. Same as part 1, with nine spheres placed inside a cube. *Suggestion:* $r \approx 0.232s$.

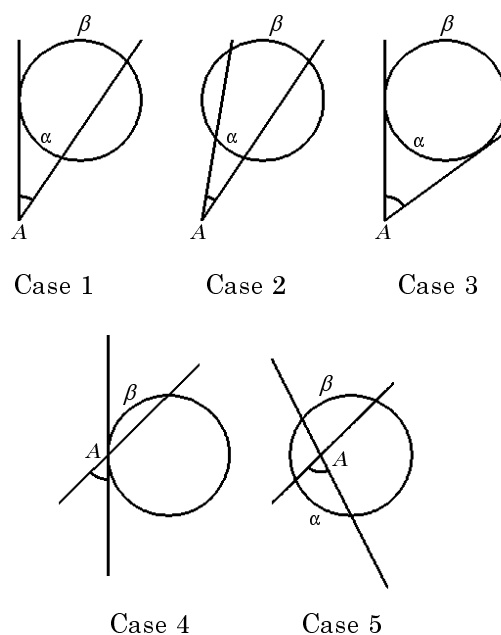
Solution, Part 1. The square's diagonal consists of four radii plus the diagonals of two squares with edge r . The equation $4r + 2r\sqrt{2} = s\sqrt{2}$ yields

$$r = \frac{\sqrt{2} - 1}{2}s \approx 0.207s. \quad \text{Part 2. } r = \frac{2\sqrt{3} - 3}{2}s \approx 0.232s. \quad \blacklozenge$$

This section concludes with two exercises on tangents and secants to circles.

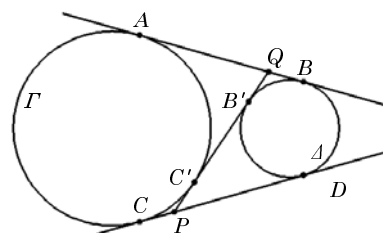
Exercise 5. Investigate the angles between two tangents to a circle, between two secants, and between a tangent and a secant. Include all five cases shown in figure 4.6.3. In each case derive a formula for the measure of $\angle A$ in terms of the measures α and β of the indicated arcs. *Suggestion:* More general cases are best derived from more special ones. You may have to consider separate cases depending on the location of the center with respect to $\angle A$. Methods from the proof of theorem 3.13.6 may help.

⁷ This word is a Latin compound: *quinque* + *uncia* = *five* + *ounce* or *twelfth*.

**Figure 4.6.3** Exercise 5

Solution. Cases 1 and 2. $m\angle A = \frac{1}{2}(\beta - \alpha)$.
 Case 3. $m\angle A = \frac{1}{2}(\beta - \alpha) = \pi - \alpha$.
 Case 4. $m\angle A = \frac{1}{2}\beta$.
 Case 5. $m\angle A = \frac{1}{2}(\alpha + \beta)$. ♦

Exercise 6. Lines $\vec{AB}$ and $\vec{CD}$ are *externally* tangent to disjoint circles Γ and Δ at A, B and C, D in figure 4.6.4. Line $\vec{B'C'}$ is *internally* tangent at C' and B' . The internal tangent intersects the others at P and Q . Prove that $AB = CD = PQ$. *Suggestion:* Methods appropriate for solving this exercise are used later to prove theorem 5.9.2.

Figure 4.6.4
Exercise 6

Solution. If the external tangents are parallel, it's obvious that $AB = CD$. Otherwise, they meet at a point X . Then $XA = XC$ and $XB = XD$, so $AB = CD$. Now,

$$\begin{aligned}
 PQ &= PB' + B'Q \\
 &= PD + B'Q \\
 &= CD - PC + B'Q \\
 &= AB - PC + B'Q \\
 &\stackrel{8}{=} AB - PC' + B'Q \\
 &= AB - (PB' - B'C') + B'Q \\
 &= AB + B'C' + B'Q - PB' \\
 &= AB + QC' - PB' \\
 &\stackrel{8}{=} AB + AQ - PB' \\
 &= AB + AQ - (PQ - QB') \\
 &= AB + AQ + QB' - PQ \\
 &\stackrel{8}{=} AB + AQ + QB - PQ \\
 &= AB + AB - PQ \\
 2PQ &= 2AB. \blacklozenge
 \end{aligned}$$

4.7 Exercises on area

Concepts

Dissecting polygonal regions
 Kites and lunes
 Interior and boundary of a polygonal region
 Open and closed subsets of a plane

This section contains exercises on polygonal regions and regions constructed from circles, emphasizing area. In exercises 3 to 8, the main technique required is *dissection*: splitting a given region into pieces that you can handle with the area theory developed in sections 3.8 and 3.14. All exercises except the last two involve area calculations. The last ones let you develop the concepts of boundary and interior of a polygonal region. Although polygonal regions are built from triangular ones, and the interior of a triangular region is a simple idea, you'll find it hard to pin down the corresponding notion for polygonal regions in general. The exercises provide a workable definition, which you'll need in chapter 8.

⁸ Equations referring to this footnote hold because of congruent tangent segments.

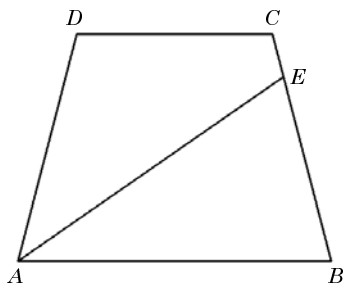


Figure 4.7.1
Exercise 1

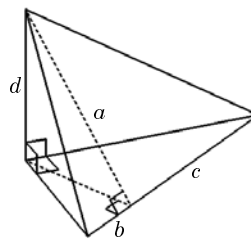


Figure 4.7S.1
Exercise 2 solution

Exercise 1. Figure 4.7.1 is an isosceles trapezoid $ABCD$ with $\vec{AB} \parallel \vec{CD}$, $AB = b_1$, $CD = b_2$, $BC = c = DA$, and a point E on $\overline{BC}$. Find $d = BE$ so that $\overline{AE}$ splits $ABCD$ into equal areas.

Solution. Let a be the trapezoid's altitude and h be the distance from A to $\overline{BC}$. Compute the area of $\triangle ABC$ two ways to get $\frac{1}{2} a b_1 = \frac{1}{2} c h$, hence $h = a b_1 / c$. Then

$$\text{area } ABCD = \frac{1}{2} a (b_1 + b_2) \qquad \text{area } \triangle ABE = \frac{1}{2} d h.$$

Setting $\frac{1}{2} \text{area } ABCD = \text{area } \triangle ABE$, you get $d = \frac{1}{2} (b_1 + b_2) c / b_1$. ♦

Exercise 2. Consider a tetrahedron with three mutually perpendicular faces. Prove that the sum of the squares of their areas equals the square of the area of the other face.

Solution. Identify one of the three mutually perpendicular faces as the pyramid's base triangle. As in Figure 4.7S.1, denote by d the corresponding altitude of the pyramid. Regard the hypotenuse of the base triangle as its base and as the base of the remaining face, an oblique triangle. Denote by a the corresponding altitude of the oblique face. The base triangle altitude is then $(a^2 - d^2)^{1/2}$. Suppose the foot of a divides the base of the oblique face into segments b and c . Then the right lateral faces have hypotenuses $(a^2 + b^2)^{1/2}$, $(a^2 + c^2)^{1/2}$ and share legs $(a^2 + b^2 - d^2)^{1/2}$, $(a^2 + c^2 - d^2)^{1/2}$ with the base triangle. Thus, the equation to be verified is

$$\begin{aligned} & [\text{area of base triangle}]^2 + [\text{area of first right lateral face}]^2 \\ & + [\text{area of second right lateral face}]^2 \\ & = [\text{area of the oblique face}]^2 \end{aligned}$$

$$\begin{aligned} & \left[\frac{1}{2} (b+c)(a^2 - d^2)^{\frac{1}{2}} \right]^2 + \left[\frac{1}{2} d(a^2 + b^2 - d^2)^{\frac{1}{2}} \right]^2 + \left[\frac{1}{2} d(a^2 + c^2 - d^2)^{\frac{1}{2}} \right]^2 \\ &= \left[\frac{1}{2} a(b+c) \right]^2. \end{aligned}$$

Expand some of the squares and multiply both sides by 4 to get the equivalent equation

$$(b+c)^2(a^2 - d^2) + d^2(a^2 + b^2 - d^2) + d^2(a^2 + c^2 - d^2) = a^2(b+c)^2.$$

Expand, gather like terms, divide by d^2 , and rearrange to get another equivalent equation:

$$(a^2 + b^2 - d^2) + (a^2 + c^2 - d^2) = (b+c)^2.$$

This one holds because the base triangle is right.⁹ ♦

Exercise 3. Using figure 4.7.2, compute the area of the regular dodecagon with radius 1.¹⁰

Solution Each small square consists of four congruent equilateral triangles and eight congruent isosceles triangles. Of these, three equilateral and six other triangles lie in the dodecagon. Thus its area is $\frac{3}{4}$ that of the large square, i.e. 3. ♦

Exercise 4. In figure 4.7.3, each edge of an outer equilateral triangle is divided in the ratio 2 to 1. Find the area of the shaded inner triangle, in terms of the edge s of the outer one. *Suggestion:* It's approximately $0.14s^2$.

Solution. Dissect the outer triangle as shown in figure 4.7S.2, forming nine component triangles with the same area. Three of them form the inner triangle, so its area is one third of the area $\frac{1}{4}s^2\sqrt{3}$ of the original triangle. ♦

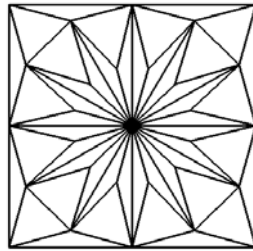


Figure 4.7.2
Kürschak's tile

⁹ For other proofs, consult Hull, Perfect, and Heading 1978; and Altshiller-Court [1935] 1964, §286.

¹⁰ This exercise is due to J. Kürschak. See Alexanderson and Seydel 1978.

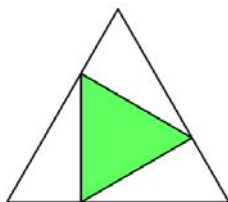


Figure 4.7.3
Exercise 4

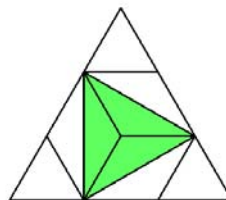


Figure 4.7S.2
Exercise 4 solution

Exercise 5. In figure 4.7.4, segments join the vertices of a square to midpoints of its sides. Prove that the quadrilateral in the middle is a square, and determine the ratio of its area to that of the outer square. Find an analogous result for a similar situation in which the outer figure is a rectangle. *Suggestion:* The only computation in your solution should be counting.

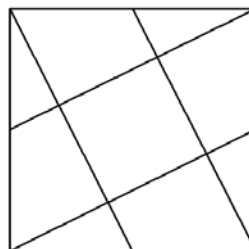
Solution. The original square is composed of twenty right triangles congruent to the small ones in figure 4.7.4. Four of them constitute the middle square, so the ratio is $\frac{4}{20} = \frac{1}{5}$.

You can make any rectangle by stretching one dimension of the original figure by a factor $k \geq 1$. The central square becomes a parallelogram. All areas are multiplied by k , so their ratios remain unchanged. ♦

Exercise 6. In figure 4.7.5, some vertices of a square have been joined to edge midpoints to form a *kite*: a quadrilateral (shaded) with two adjacent pairs of congruent edges. Find its area, in terms of the edge s of the square. *Suggestion:* It's about $0.27s^2$.

Solution. As shown in figure 4.7S.3a, $\vec{QR'}$ intersects $\vec{OQ'}$ at a point S on diagonal $\vec{PR}$. Let T be the foot of the perpendicular to $\vec{OP}$ through S , so that $ST = TP$. Further, $\triangle STO \sim \triangle Q'O'O$, so $OT = \frac{1}{2}ST = \frac{1}{2}TP$,

Figure 4.7.4
Exercise 5



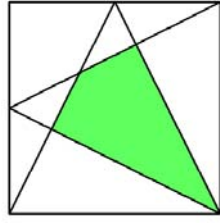


Figure 4.7.5
Exercise 6: kite

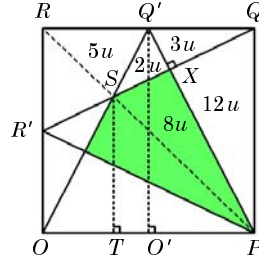


Figure 4.7S.3
Exercise 6 solution

hence $OT = \frac{1}{3}s$ and $TP = \frac{2}{3}s$. Now notice that $P\vec{Q}' \perp Q\vec{R}'$; call their intersection X . Then $\Delta QXQ' \sim \Delta QRR'$, so $QX = 2(Q'X)$ and

$$\frac{1}{4}s^2 = (QQ')^2 = (QX)^2 + (Q'X)^2 = 5(Q'X)^2,$$

so $(Q'X)^2 = \frac{1}{20}s^2$ and $(QX)^2 = \frac{1}{5}s^2$. Also,

$$\frac{SQ'}{OS} = \frac{TO'}{OT} = \frac{s/6}{s/3} = \frac{1}{2}$$

$$\begin{aligned} (SQ')^2 &= \frac{1}{4}(OS)^2 = \frac{1}{4}[(OT)^2 + (TS)^2] = \frac{1}{4}[(OT)^2 + [2(OT)]^2] \\ &= \frac{1}{4}[5(OT)^2] = \frac{5}{4}\left(\frac{1}{3}s\right)^2 = \frac{5}{36}s^2 \end{aligned}$$

$$(SX)^2 = (SQ')^2 - (Q'X)^2 = \frac{5}{36}s^2 - \frac{1}{20}s^2 = \frac{4}{45}s^2$$

$$\frac{(SX)^2}{(QX)^2} = \frac{\frac{4}{45}}{\frac{1}{5}} = \frac{4}{9}$$

$$\frac{SX}{QX} = \frac{2}{3}.$$

You've now enough information to determine the ratios of the areas of all triangles in figure 4.7S.3. Assign $\Delta QQ'X$ the area $3u$. You can determine the unit u later. Since $SX/XQ = \frac{2}{3}$, $\Delta SXQ'$ has area $2u$. Since $\Delta QSQ'$ and $\Delta Q'SR$ have the same altitude and congruent bases $\overline{RQ'}$ and $\overline{QQ'}$, they have the same area $5u$. Triangles $\Delta QXQ'$ and ΔQXP have bases $\overline{XQ'}$ and $\overline{XP}$ and the same altitude. Since $XP = 2(XQ) = 4(XQ')$, ΔQXP has area $12u$. For the same reasons, ΔPXS has area $8u$. Thus, $3u + 2u + 5u + 12u + 8u = 30u = \frac{1}{2}s^2$, the area of ΔPQR . It follows that $u = \frac{1}{60}s^2$. The shaded area is twice that of ΔPXS : $2(8u) = \frac{4}{15}s^2$. ♦

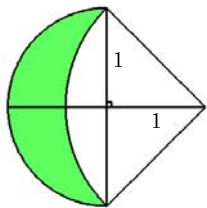


Figure 4.7.6
Exercise 7

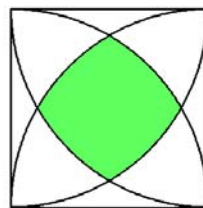


Figure 4.7.7
Exercise 8

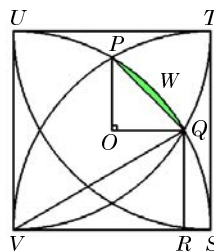
Exercise 7. Compute the area of the shaded region in figure 4.7.6. Called a *lune*,¹¹ it lies “between” arcs of two circles with radius 1. The solution may surprise you.

Solution. The shaded area is half that of a circle with radius r , plus the area of two isosceles right triangles with legs r , minus a quarter of the area of a circle with radius $r\sqrt{2}$: $\frac{1}{2}\pi r^2 + 2(\frac{1}{2}r^2) - \frac{1}{4}\pi(r\sqrt{2})^2 = r^2$. ♦

Exercise 8. Compute the area of the shaded region in figure 4.7.7. The length of an edge of the square is 1. *Suggestion:* This exercise is related to exercise 3; the area is approximately 0.32.

Solution. (See figure 4.7S.4.) First, $QR = \frac{1}{2}$ and $VQ = 1$, so $m\angle QVR = 30^\circ$. The same argument applies to $\angle PVU$, so $m\angle PVQ = 90^\circ - 2 \cdot 30^\circ = 30^\circ$. The area of sector VPQ is $\frac{1}{12}$ of that of a circle with radius 1, i.e. $\frac{1}{12}\pi$. The area of $\triangle VPQ$ is $\frac{1}{12}$ of that of a regular dodecagon with radius 1, which is $\frac{3}{12}$, according to exercise 3. The area a of the shaded region between segment $\overline{PQ}$ and arc $\widehat{PWQ}$ is thus $\frac{1}{12}\pi - \frac{1}{4}$. Since $OQ = VR - \frac{1}{2} = \frac{1}{2}\sqrt{3} - \frac{1}{2}$, the area of $\triangle OPQ$ is $b = \frac{1}{2}(OQ)^2 = \frac{1}{4}(2 - \sqrt{3})$. The desired area is

Figure 4.7S.4
Exercise 8 solution



¹¹ From the Latin word *luna* for *moon*.

$$4(a + b) = 4 \left(\frac{2 - \sqrt{3}}{4} + \frac{\pi}{12} - \frac{1}{4} \right) = 1 - \sqrt{3} + \frac{1}{3} \pi \approx 0.32. \blacklozenge$$

The *interior* of a triangle is the intersection of the interiors of any two, hence all three, of its angles. Even though a polygonal region Π is the union of regions corresponding to triangles T_1 to T_n , and the notion of the interior of each T_i is clear, that of the interior of Π is surprisingly hard to make precise. The section's concluding exercises will force you to confront that difficulty, and will provide a definition you can use later, in chapter 8. For now, trust your intuition to answer correctly whether a given point P is interior to Π . The problem is to describe in words how your intuition works.

Exercise 9, Part 1. Is the interior of Π always the union of the interiors of T_1 to T_n ? Using $n = 2$, present an example where this is true and a counterexample where it's not. *Suggestion:* Your counterexample will probably involve triangles that intersect along an edge.

Part 2. Does the interior of Π ever consist of the union of the interiors of T_1 to T_n and the points on their common edges? Using $n = 2$, present an example where this is true and a counterexample where it's not.¹² *Suggestion:* Your counterexample will probably involve a shared vertex.

Part 3. Does the interior of Π always consist of the union of the interiors of T_1 to T_n and the points interior to common edges? Using $n = 3$ and triangles T_i that intersect, present an example where this is true and a counterexample where it's not.

Solution. *Parts 1 and 2* are simple. *Part 3.* For an example, consider three equilateral triangles that share a common vertex; for a counterexample, three triangles with 120° angles sharing a common vertex. $\blacklozenge$

You should be pleased that your intuition works in complicated situations, but troubled that it's hard to describe. You *will* need a precise definition of the interior of a polygonal region Π in a plane ε . You may have noticed that part of the difficulty lies in deciding whether some points are on its boundary. In fact, the problems of defining interior and boundary are equivalent, because the interior of Π should be the set of its nonboundary points, and its boundary should be the set of its noninterior points. Here's a precise definition: a point P lies on the *boundary* of Π just in case

for every triangle T in ε , if P is interior to T , then there exist points U and V interior to T such that U lies in Π but V does not.

¹² Does it make any difference whether you define *polygonal region* as in section 3.8, note 13, so that the T_i intersect, if at all, only along *entire* edges?

You can use this to define the *interior* of Π precisely, as the set of all non-boundary points in Π .

Exercise 10, Part 1. Show that you could replace the word *triangle* in this definition by *circle* or *square*.

Part 2. Show that every boundary point of Π lies on an edge of one of the triangles T_i .

Part 3. Show that a point P lies in the interior of Π if and only if P is interior to some triangle, all of whose interior points belong to Π .

Part 4. First, a definition: A set Γ of points in a plane ε is called *open* in ε if every point P of Γ is interior to some triangle in ε , all of whose interior points belong to Γ . Part 3 showed that the interior of a polygonal region Π in ε is open in ε . Now show that the boundary of Π contains no set that's open in ε .

Part 5. A set Σ of points in ε is called *closed* in ε if its relative complement—the set of points in ε but *not* in Σ' —is open. Show that Π is closed.

Solution, Part 1. Consider a figure consisting of a triangle inside a square inside a circle inside a triangle.

Part 2. A boundary point of Π must lie on some triangle T_i or in its interior. If it were interior to T_i , then some point V interior to T_i , hence in the corresponding region, would have to lie outside Π —contradiction!

Part 3. Suppose P belongs to the interior of Π . Then P is interior to some triangle T_i but by part 2 cannot lie on an edge of T_i . Find a triangle T such that P is interior to T and all points of T are interior to T_i . Then all points interior to T are interior to Π . The *only if* implication of part 3 has been proved. Its converse follows directly from the definition of interior.

Part 4. Suppose the boundary contained an open set Γ . Consider any point $P \in \Gamma$; it would be interior to some triangle T all of whose interior points belong to Γ . But P would also lie on the boundary of Π , hence some point V interior to T must *not* belong to Π , and V would therefore *not* belong to Γ —contradiction!

Part 5. Suppose P lies outside Π . Find a triangle Σ_1 such that P is interior to Σ_1 and all points interior to Σ_1 lie outside T_1 . Find a triangle Σ_2 such that P is interior to Σ_2 and all points interior to Σ_2 lie interior to Σ_1 and outside T_2 . Continue until you've found a triangle Σ_n such that P is interior to Σ_n and all points interior to Σ_n lie interior to Σ_{n-1} and outside T_n . ♦

4.8 Exercises on volume

Concepts

Dissecting volumes
 Prismoids and antiprisms
 Volumes of spheres and spherical caps
 Interior and boundary of a polyhedral region
 Open and closed point sets

Like most exercises in the previous section, exercises 1 and 2 demonstrate *dissection*: splitting a given region into pieces that you can handle with the volume theory developed in sections 3.10 and 3.14. Exercise 3 provides the proof for the last result mentioned in section 3.14: the formula for the volume of a sphere. Exercise 4 considers some volumes related to that of a sphere. Exercise 5 extends to three dimensions the discussion of interior and boundary points and open and closed sets presented for plane geometry in exercises 4.7.9 and 4.7.10.

Exercise 1. Make a polyhedron—called a *prismoid*—with two dissimilar rectangular faces $ABCD$ and $A'B'C'D'$ in parallel planes, so that pairs of corresponding edges lie in parallel lines. The remaining four lateral faces are trapezoids whose bases are corresponding edges of the rectangles. Find its volume in terms of the rectangles' edges a, b and a', b' and the distance h between the base planes. *Suggestion:* If $a, b, a', b', h = 2, 3, 4, 5, 6$ then the volume is 74.

Solution. Let h be the distance between the base planes. Select any inside point O on the plane ε midway between the bases. Use diagonals to subdivide each lateral face into two large triangles. Use ε to subdivide each large triangle into a small triangle and a small trapezoid. Consider the pyramids with apex O whose bases are the large and small triangles. The area of each small triangle is $\frac{1}{4}$ that of the corresponding large triangle, so the volume of each small triangular pyramid is $\frac{1}{4}$ that of the corresponding large triangular pyramid. Now reconsider a small triangular pyramid. Make its face on ε the base, so that the corresponding altitude is $\frac{1}{2}h$. The total volume of the small triangular pyramids is thus $\frac{1}{3} \cdot \frac{1}{2}h$ times the total area x of their faces in plane ε . But x is the area of a rectangle whose edges are the averages of the corresponding base edges:

$$x = \frac{1}{2}(a + b) \cdot \frac{1}{2}(a' + b').$$

Thus, the total volume of the large triangular pyramids is $4 \cdot \frac{1}{3} \cdot \frac{1}{2}hx$. The remainder of the prismoid consists of pyramids with apex O whose bases

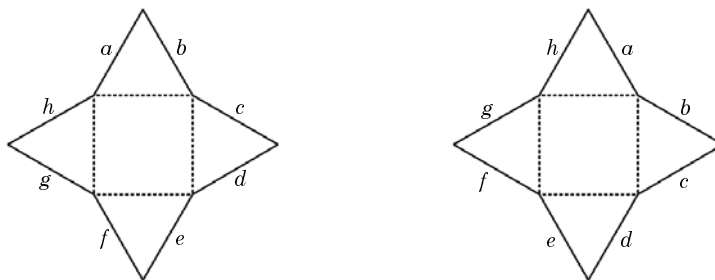


Figure 4.8.1 Antiprism cutouts for exercise 2

are the two rectangles. Their volumes are $\frac{1}{3} \cdot \frac{1}{2} h a b$ and $\frac{1}{3} \cdot \frac{1}{2} h a' b'$. Adding all these components yields the total volume¹³

$$\frac{1}{6} (2 a b + a' b + a b' + 2 a' b') h. \blacklozenge$$

Exercise 2. Make cutouts like the two parts of figure 4.8.1, so that all edges a to h and those of the internal squares have the same length s . Make a polyhedron—called a *square antiprism*—by folding along the dotted lines and taping similarly lettered edges together. Find its volume. *Suggestion:* Dissect the antiprism into several pyramids. If $s = 1$, their total volume is approximately 0.957.

Solution. Let A and C be the centers of the squares. When you've assembled the antiprism, the squares lie in parallel planes perpendicular to $\overline{AC}$. Let B be the midpoint of an edge of the square with center A , and E be the other square's vertex nearest B . Then $AB = \frac{1}{2} s$, $BE = \frac{1}{2} s \sqrt{3}$ because it's the altitude of an equilateral triangle with edge s , and $CE = \frac{1}{2} s \sqrt{2}$ because it's half the diagonal of a square with edge s . Let D be the foot of the perpendicular from B to the other plane, so that D falls in $\overline{CE}$. Then

$$DE = CE - AB = \frac{1}{2} (\sqrt{2} - 1) s$$

$$AC = BD = \sqrt{(BE)^2 - (DE)^2} = s/2^{1/4}.$$

Now consider the trapezoid $ABEC$ shown in detail in figure 4.8S.1. Let M and G be the midpoints of $\overline{AC}$ and $\overline{BE}$, so that $\overline{MG} \parallel \overline{AB}$. Let F be the foot of the perpendicular to $\overline{BE}$ through M , and H be the intersection of $\overline{MG}$ and $\overline{BD}$. Then

¹³ See Bradley 1979 for another derivation and for a generalization of this argument.

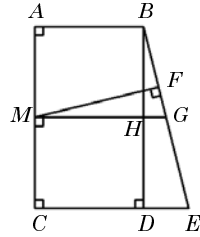


Figure 4.8S.1
Exercise 2 solution

$$BG = \frac{1}{2} BE = \frac{1}{4} s \sqrt{3}$$

$$MG = \frac{1}{2} (AB + CE) = \frac{1}{4} (1 + \sqrt{2}) s.$$

By similarity, $MF/MG = BH/BG$ and

$$MF = \frac{(BH)(MG)}{BG} = \frac{(AC)(MG)}{2(BG)} = \frac{1 + \sqrt{2}}{2^{\frac{4}{3}} \sqrt{2} \sqrt{3}} s.$$

The antiprism is composed of ten pyramids with apex M , one based on each of its faces. Two are square, with base areas s^2 and altitudes $\frac{1}{2} AC = s/2^{5/4}$. Their total volume is

$$v = 2 \cdot \frac{1}{3} \cdot \frac{s}{2^{5/4}} \cdot s^2 = \frac{s^3}{3^{\frac{4}{3}} \sqrt{2}}.$$

The other eight are triangular. Their equilateral bases have area $\frac{1}{4} s^2 \sqrt{3}$ and altitude MF ; their total volume is

$$w = 8 \cdot \frac{1}{3} \cdot \frac{1 + \sqrt{2}}{2^{\frac{4}{3}} \sqrt{2} \sqrt{3}} s \cdot \frac{1}{4} s^2 \sqrt{3} = \frac{1 + \sqrt{2}}{3^{\frac{4}{3}} \sqrt{2}} s^3.$$

The total volume of the antiprism is thus

$$v + w = \frac{\sqrt[4]{8} + \sqrt[4]{2}}{3} s^3 \approx 0.957 s^3. \blacklozenge$$

Exercise 3. A plane through a point O divides a sphere with center O and radius r into two hemispheres. On their common circular base, construct a cylinder with altitude r . Using Cavalieri's axiom, show that the volume of the region between the cylinder and one of the hemispheres is the same as that of a certain cone. Apply the volume formulas for the cylinder and cone to derive the formula for the volume of the sphere.

Solution. Intersections of the region between the cylinder and hemisphere with parallel planes at distance h from the base are the regions between

concentric circles with radii r and $\sqrt{r^2 - h^2}$, so they have area πh^2 . That's the same as the area of the corresponding intersection with the right cone with apex O and the same altitude and base as the cylinder. Therefore,

$$\text{volume of cylinder} - \frac{1}{2} \text{volume of sphere} = \text{volume of cone}$$

$$\begin{aligned} \text{volume of sphere} &= 2(\text{volume of cylinder} - \text{volume of cone}) \\ &= 2(\pi r^2 \cdot r - \frac{1}{3} \pi r^2 \cdot r) = \frac{4}{3} \pi r^3. \blacklozenge \end{aligned}$$

Exercise 4, Part 1. Let $0 \leq h < r$. A plane at distance h from O divides the sphere with center O and radius r into two parts. The smaller is called a *spherical cap*. Without using calculus, find a formula for its volume. *Suggestion:* If $r = 1$ and $h = \frac{1}{2}$, the cap's volume is about 0.65.

Part 2. Without using calculus, find the volume of material removed from of a hole with radius $\frac{1}{2}$ drilled through the center of a sphere with radius 1. *Suggestion:* The hole's volume is about 0.47.

Solution, Part 1. Use the method of exercise 3 between the planes at distance h and r :

$$\begin{aligned} &\text{volume of the cylinder between these planes} - \text{the required volume} \\ &= \text{volume of the region of the cone between these planes} \\ &= \text{volume of the cone with apex } O \\ &\quad \text{and altitude and base radius } r \\ &\quad - \text{volume of the cone with apex } O \\ &\quad \text{and altitude and base radius } h \end{aligned}$$

$$\begin{aligned} \text{required volume} &= \pi r^2(r - h) - \frac{1}{3} \pi r^2 r + \frac{1}{3} \pi h^2 h \\ &= \frac{2}{3} \pi r^3 - \pi r^2 h + \frac{1}{3} \pi h^3. \end{aligned}$$

Part 2. The hole consists of two caps at distance $h = \frac{1}{2} \sqrt{3}$ from the center of the sphere with radius $r = 1$, and a cylinder with radius $s = \frac{1}{2}$ and altitude $k = 2h$. Its volume is

$$2(\frac{2}{3} \pi r^3 - \pi r^2 h + \frac{1}{3} \pi h^3) + \pi s^2(2h) = (\frac{4}{3} - \frac{1}{2} \sqrt{3}) \pi. \blacklozenge$$

Exercise 5. Construct and solve a three-dimensional analog of exercises 4.7.9 and 4.7.10. This must include precise definitions of the interior and boundary of a polyhedral region, and of open and closed sets in three-dimensional space.

Solution. Parts 1 to 3 of exercise 4.7.9 go easily, with an extension to the question whether the interior of polyhedral region Π always consists of the union of the interiors of tetrahedral regions T_1 to T_n and the points interior to common faces. The precise definition of the *boundary* of a polyhedral region is the same as that for the boundary of a polygonal one, except you must change the word *triangle* to *tetrahedron*. The definition

of *interior* is the same. The extension of part 1 of exercise 4.7.10 should read, *Show that you could replace the word “tetrahedron” in this definition by “sphere” or “cube”.* To construct the analogs of parts 2 and 3 of exercise 4.7.10, change the word *triangle* to *tetrahedron*. The definition of *open* set is the same, except you must make that same change. The definition of *closed* is the same, and the analogs of parts 4 and 5 of exercise 4.7.10 are identical to their two-dimensional counterparts. The proofs of all these results are related in the same way to the two-dimensional ones. ♦

4.9 Exercises on circles and spheres, part 2

Concepts

$\cos 36^\circ$

Archimedes' approximation to π

Great circles and antipodal points on a sphere

Measuring great circular arcs and angles between them

Spherical triangles

Trigonometry of right spherical triangles

Curved area of cylinders, cones, spheres, and spherical triangles

This section contains some specialized problems on circles and spheres. Exercise 1 adds a new angle, 36° , to those for which elementary geometry yields the values of the trigonometric functions. Later exercises work out Archimedes' approximation to π , the beginnings of spherical trigonometry, and some example curved area calculations.

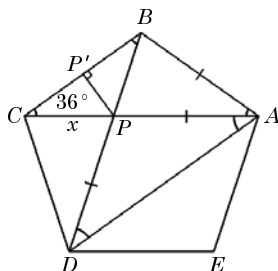
Exercise 1, Part 1. Consider a regular pentagon $ABCDE$ with edge 1. Show that two diagonals $\overline{AC}$ and $\overline{BD}$ meet at the point P for which $APDE$ is a parallelogram. Let $x = PC$. Use similar and congruent triangles to show that

$$x = \frac{1}{1+x}.$$

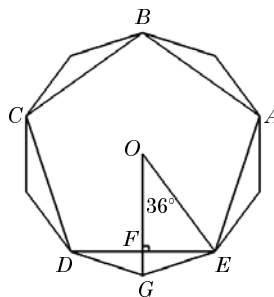
Compute the value of x . Finally, use x and $\triangle BCP$ to compute $\cos 36^\circ$.

Part 2. Consider a regular pentagon, hexagon, and decagon with the same radius. Show that you can make a right triangle from segments congruent to their edges.¹⁴

¹⁴ Part 2 is proposition 10 of book XIII of Euclid's *Elements* ([1908] 1956).

**Figure 4.9S.1**

Exercise 1 solution, part 1

**Figure 4.9S.2**

Exercise 1 solution, part 2

Solution, Part 1. Consider figure 4.9S.1. The proportionality

$$x = \frac{x}{1} = \frac{PC}{BC} = \frac{AP}{AD} = \frac{1}{1+x}$$

implies $x(x+1) = 1$, i.e. $x^2 + x - 1 = 0$, hence $x = \frac{1}{2}(-1 + \sqrt{5})$. Let P' be the midpoint of $\overline{BC}$. Then

$$\cos 36^\circ = \frac{CP'}{CP} = \frac{\frac{1}{2}}{x} = \frac{1+\sqrt{5}}{4} \approx 0.8090.$$

Part 2. Consider a regular pentagon $ABCDE$ with center O and radius 1, and let F be the midpoint of edge $\overline{DE}$, as in figure 4.9S.2. Find point G on $\overline{OF}$ such that $OG = 1$, so that $\overline{GE}$ is an edge of the regular decagon whose vertices include A to E . Then $OF = \cos 36^\circ$, $FE = \sin 36^\circ$, $DE = 2 \sin 36^\circ$, $FG = 1 - \cos 36^\circ$, and $(GE)^2 = (FE)^2 + (FG)^2 = 2 - 2 \cos 36^\circ$. It's easy to see that a regular hexagon with radius 1 has edge 1, so the desired result is $(GE)^2 + 1 = (DE)^2$, i.e. $3 - 2 \cos 36^\circ = 4 - 4 \cos^2 36^\circ$. You can verify this using the value $\cos 36^\circ$ from part 1. ♦

In exercise 2 you'll carry out Archimedes' approximation¹⁵ to π as described earlier in section 3.14. You'll see that if you can perform the familiar arithmetic operations and calculate square roots, then you can compute π as accurately as you might wish. Exercise 3, a sidelight, shows that our idea of circle geometry and π might be quite different if we considered only the geometry of the surface of the earth.

Exercise 2. Consider regular polygons with n edges of length s_n and t_n , inscribed in and circumscribed about a circle with radius 1. Thus,

¹⁵ See van der Waerden 1963, chapter 7.

$\frac{1}{2} n s_n < \pi < \frac{1}{2} n t_n$. Compute s_3 and t_3 . Find formulas for t_n and s_{2n} in terms of s_n . Use these formulas to compute $\frac{1}{2} n s_n$ and the error bound $\varepsilon_n = \frac{1}{2} n t_n - \frac{1}{2} n s_n$ for $n = 6, 12, 24, 48$, and 96 . What's the resulting approximation of π and how accurate is it guaranteed to be? This is Archimedes' result. How big must n be to guarantee that the error is at most 10^{-6} ?

Solution. First, $s_3 = \sqrt{3}$ and $t_3 = 2\sqrt{3}$. Further,

$$t_n = \frac{2s_n}{\sqrt{4 - s_n^2}} \quad s_{2n} = \sqrt{2 - \sqrt{4 - s_n^2}}.$$

Applying these formulas repeatedly yields

n	$\frac{1}{2} n s_n$	ε_n
3	2.5980762	2.5980762
6	3.0000000	0.4641016
12	3.1058285	0.1095618
24	3.1326286	0.0270313
48	3.1393502	0.0067360
96	3.1410320	0.0016827
$\vdots$	$\vdots$	$\vdots$
6144	3.1415925	0.0000004

Caution! This output required IEEE double precision arithmetic. With single precision, π doesn't fall between the computed values of $\frac{1}{2} n s_n$ and $\frac{1}{2} n t_n$ for $n = 6144$. ♦

Exercise 3. Consider a sphere Σ with center O and radius R . A plane ε not through O intersects Σ in a circle Γ with center P and radius r . The perpendicular to ε through P meets Σ at a point Q on the side of ε opposite O . Let X be any point on Γ . You could call the length r' of arc $\widehat{QX}$ the “radius of the circle measured on the sphere” and the ratio of the circumference of Γ to $2r'$ as an “approximation to π measured on Σ .” This approximation depends on the circle. What distance r' on the earth (a sphere) gives the approximation $\pi \approx 3$?

Solution. Let $\theta = m\angle QOX$. Then

$$3 = \frac{2\pi r}{2r'} = \frac{\pi r}{r'} = \frac{\pi R \sin \theta}{\frac{\theta}{360^\circ} \cdot 2\pi R} = \frac{180^\circ \sin \theta}{\theta}.$$

A glance at the graph of the function $(180^\circ \sin \theta)/\theta - 3$ suggests its root: $\theta = 30^\circ$. The earth's radius R is about 3959 miles, so $r' = (\theta/360^\circ) \cdot 2\pi R = \frac{1}{6} \pi R \approx 2073$ miles. ♦

In exercise 4 you'll work out the beginning of spherical trigonometry. Exercise 8 and some exercises in section 5.11 pursue the subject further. First, you must understand what a spherical triangle is and how you measure it. A *great circle* on a sphere Σ is its intersection with a plane through its center O . Any two points P and Q on Σ that aren't collinear with O determine a great circle Γ . Using calculus, it's shown that the shortest path between these points that lies entirely in Σ is the minor arc they define on Γ . (Two *antipodal* points—distinct points on Σ collinear with O —determine many great circles, and on any of these, either semicircle is a shortest path between them.) Thus, to develop geometry of the sphere analogous to that of the plane, it's useful to make arcs of great circles play the role of line segments. Instead of basing spherical geometry on the *length* of these arcs though, it's simpler to use their *degree measures*. That lets you state many results without mentioning the sphere's radius.

Consider points A , B , and C on Σ that aren't coplanar with O . You can define the *angle* between the intersecting great circle arcs $\widehat{AB}$ and $\widehat{BC}$ as the dihedral angle formed by the sides of $\widehat{OB}$ in the planes ABO and BCO in which the arcs lie. These points also define a *spherical triangle* $\triangle ABC$ whose edges are the minor arcs $\widehat{AB}$, $\widehat{BC}$, and $\widehat{CA}$ of the great circles. (This is ambiguous if two of these points are antipodal. You must use more detailed language to handle that case.) A spherical triangle $\triangle ABC$ has six measurements:

$$\begin{array}{lll} \text{m}\angle A & \text{m}\angle B & \text{m}\angle C \\ a = \text{m}\widehat{BC} & b = \text{m}\widehat{CA} & c = \text{m}\widehat{AB} . \end{array}$$

Exercise 4 develops equations for these measurements in a *right* spherical triangle with $\text{m}\angle C = 90^\circ$.

Exercise 4. Prove that in this right spherical $\triangle ABC$, $\cos a \cos b = \cos c$ and

$$\begin{array}{ll} \sin \text{m}\angle A = \frac{\sin a}{\sin c} & \cos \text{m}\angle A = \frac{\cos a \sin b}{\sin c} \\ \sin \text{m}\angle B = \frac{\sin b}{\sin c} & \cos \text{m}\angle B = \frac{\cos b \sin a}{\sin c} . \end{array}$$

Construct examples like those in exercise 4.5.4 to show how to find any three of $\text{m}\angle A$, $\text{m}\angle B$, a , b , and c given the remaining two. *Suggestion:* You may find exercise 4.3.4 helpful.

Solution. Denote the sphere's radius by r . Find the foot F of the perpendicular to plane OBC through A and the foot P of the perpendicular to $\widehat{OB}$ through F . By exercise 4.3.4, $\vec{AP} \perp \vec{OB}$. From $\triangle PFA$, you get $\sin \text{m}\angle B = AF/AP$. From $\triangle OPA$, you get $AP/r = \sin c$. From $\triangle OFA$, you get $AF/r = \sin b$. These equations entail $\sin \text{m}\angle B = \sin b/\sin c$. You can derive $\sin \text{m}\angle A = \sin a/\sin c$ similarly.

From $\triangle OPA$, you get $\cos c = OP/r$. From $\triangle OPF$, you get $OP/OF = \cos a$. From $\triangle OFA$, you get $OF/r = \cos b$. These equations entail

$$\cos a = \cos b \cos c. \quad (*)$$

From $\triangle PFA$, you get $\cos m\angle B = PF/AP$. From $\triangle OPA$, you get $AP/r = \sin c$. From $\triangle OPF$, you get $PF/OF = \sin a$. These equations entail $\cos m\angle B = \sin a \cos b / \sin c$. You can derive $\cos m\angle A = \cos a \sin b / \sin c$ similarly.

From equation (*) and the four equations displayed in the statement of this exercise you can easily derive the values of any three of the measurements from those of the remaining two. ♦

Exercises 5 to 7 develop informally a theory for the curved areas of cylinders, cones, and spheres. No axiomatic system in this book supports this; it's more delicate than the notions discussed in chapter 3. You'll derive various formulas here by means that seem reasonable, some by more than one route. But you'll find no compelling argument that different derivations will *always* yield the same result. A broader, consistent, and more robust theory can be based on calculus.

Exercise 5. You can “unroll” the curved area of a cylinder or cone so it lies flat. Whatever the curved area is, it should be the same as the flat area. Derive formulas for these curved areas in terms of the base radius r and altitude h .

Solution. The cylindrical area unrolls into a rectangle with edge lengths $2\pi r$ and h , so it must be $2\pi rh$. The conical area unrolls into a circular sector with radius $R = \sqrt{r^2 + h^2}$ and arc length $2\pi r$, so it must be

$$\frac{2\pi r}{2\pi R} \pi R^2 = \pi r R = \pi r \sqrt{r^2 + h^2}. \quad \blacklozenge$$

Archimedes reasoned¹⁶ that if he should divide the interior of a sphere with center O and radius r into very many regions like pyramids with apex O but with spherical triangular “bases,” then their total volume $\frac{4}{3}\pi r^3$ would be approximately the sum of the volumes of the corresponding triangular pyramids, which is $\frac{1}{3}r$ times their total base area. That area would be approximately the same as the curved area a of the sphere. Thus, he argued, $\frac{4}{3}\pi r^3 = \frac{1}{3}ra$, hence $a = 4\pi r^2$, provided the errors in the two approximations both disappear as the number of pyramids increases.

Exercise 6. Present two more arguments for this formula. One, used in elementary texts now, is based on the difference of the volumes of two spheres

¹⁶ See van der Waerden 1963, Chapter 7.

of nearly equal radii. Archimedes gave the other,¹² which he based on polygonal approximations to a circle and on the curved area formula for a cone.

Solution. If the radii are r and $r + h$, then the volume of the region between the two spheres is $\frac{4}{3}\pi(r + h)^3 - \frac{4}{3}\pi r^3 \approx ah$. This gives $4\pi r^2 h + 4\pi r h^2 + \frac{4}{3}\pi h^3 \approx ah$, hence $a \approx 4\pi r^2 + 4\pi h^2(r + \frac{1}{3}h)$. As h approaches zero, the right hand side approaches $4\pi r^2$, so a should be $4\pi r^2$. ♦

Exercise 7, Part 1. Derive a formula for the curved area of a cap of a sphere with radius r formed by a plane at distance $h < r$ from its center.

Part 2. Derive a formula for the curved area of the region of a sphere between two parallel planes at distances h and $h + w < r$ from its center.

Part 3. The state of Wyoming, a rectangle on a Mercator map, extends between latitudes 41° N and 45° N and longitudes $104^\circ 3'$ W and $111^\circ 3'$ W. What's its area? You may assume the earth is a sphere with radius $r = 3959$ miles.¹⁷

Part 4. Colorado, also rectangular, borders Wyoming to the south. Its edge arcs have exactly the same degree measures as Wyoming's, but it's shifted 2° eastward. What's its area? Why don't these areas agree exactly with the official figures in a reference book?

Solution, Part 1. Imagine a cone with apex at the center of the earth, using the cap's base. Its volume is $\frac{1}{3}\pi(r^2 - h^2)h$. By the solution to exercise 4.8.4, the volume of the region Γ consisting of cap and cone is

$$\frac{2}{3}\pi r^3 - \pi r^2 h + \frac{1}{3}\pi h^3 + \frac{1}{3}\pi(r^2 - h^2)h = \frac{2}{3}\pi r^2(r - h).$$

The ratio of the volume of Γ to that of the sphere is

$$\frac{\frac{2}{3}\pi r^2(r - h)}{\frac{4}{3}\pi r^3} = \frac{1}{2} \frac{r - h}{r} = \frac{1}{2} \left(1 - \frac{h}{r}\right).$$

This should be the same as the ratio of the curved areas of the cap and the sphere, so the cap's area should be

$$\frac{1}{2} \left(1 - \frac{h}{r}\right) (4\pi r^2) = 2\pi r^2 - 2\pi r h.$$

Part 2. The required area is the difference of the areas of caps at distances h and $h + w$, so it's $[2\pi r^2 - 2\pi r h] - [2\pi r^2 - 2\pi r(h + w)] = 2\pi r w$.

Part 3. Equations $h = r \sin 41^\circ$ and $h + w = r \sin 45^\circ$ determine $w = r(\sin 45^\circ - \sin 41^\circ)$, so Wyoming's area is

¹⁷ $1^\circ = 60' = \text{sixty minutes}$; $1' = 60'' = \text{sixty seconds}$.

$$\frac{111^\circ 3' - 104^\circ 3'}{360^\circ} (2\pi r^2) (\sin 45^\circ - \sin 41^\circ) \approx 97,750 \text{ mi}^2.$$

Part 4. Colorado's is

$$\frac{\sin 41^\circ - \sin 37^\circ}{\sin 45^\circ - \sin 41^\circ} \cdot (\text{Wyoming's area}) \approx 103,900 \text{ mi}^2.$$

These area figures aren't exact because the earth isn't exactly a sphere and surveyors don't necessarily follow exactly the parallels and meridians of latitude and longitude. ♦

Exercise 8. Consider a spherical $\triangle ABC$ on a sphere Σ with center O and radius r . Planes AOB and AOC intersect the sphere in two great circles that divide it into four regions called *lunes*. The A lune is the one that contains arc $\widehat{BC}$. What's its area, in terms of $m\angle A$ and r ? You can handle the B and C lunes similarly. Show that the sum of the areas of these three lunes is half that of the sphere plus twice the area of spherical $\triangle ABC$. Derive a formula for the area of the spherical triangle in terms of r and the sum m of the measures of its angles. Finally, prove that $180^\circ < m < 540^\circ$. These inequalities constitute the *spherical triangle sum theorem*.

Solution. By observing how the great circles subdivide a model of a sphere you can verify that the sum of the lunes' areas is half the sphere's plus twice the spherical triangle's. The A lune has area

$$\frac{m\angle A}{360^\circ} \cdot 4\pi r^2 = \frac{m\angle A}{90^\circ} \cdot \pi r^2.$$

Therefore,

$$\frac{m\angle A + m\angle B + m\angle C}{90^\circ} \cdot \pi r^2 = \frac{1}{2} \cdot 4\pi r^2 + 2 \text{ a } \triangle ABC$$

$$\text{a } \triangle ABC = \left(\frac{m\angle A + m\angle B + m\angle C}{180^\circ} - 1 \right) \pi r^2.$$

Finally, $180^\circ < m < 540^\circ$ because the area must be positive and less than that of a hemisphere. ♦

4.10 Exercises on coordinate geometry

Concepts

Constructing general, robust conceptual tool kits that are easy to use
Analytic geometry tool kit

Exercises 1 to 6 constitute a review of elementary three-dimensional coordinate geometry. Solving them, you'll construct a tool kit of analytic formulas and methods corresponding to many of the synthetic concepts in chapter 3. Here, phrases such as *point P* , *line g* , and *plane ε* mean *coordinates of P* , *linear parametric equations for g* , and *linear equation or linear parametric equations for ε* . Exercise 7 adapts the tools to plane geometry. For this tool kit, you should use the theory and notation summarized in appendix C for vector algebra and systems of linear equations. You may also want to consult a standard analytic geometry text.¹⁸ For each part of these exercises, you should describe the corresponding method in detail and provide enough examples to demonstrate and check it in all relevant cases.

Guidelines for constructing conceptual tool kits have emerged from recent software engineering projects. They apply to mathematical as well as programming practice. Construct a tool kit for a whole field of applications, but independent of any particular one. Make your tools general, so that you can use them for *every* problem you encounter in that field. That way, you'll retain familiarity with them. Employ uniform methods across the entire tool kit, so it's easy to use and document. Write your documentation intending that your tools be used by others with different backgrounds, and by yourself later when you don't remember details of the underlying theory. Finally, develop your tool kit ahead of time, when you're concentrating on its theory and all its details. You won't be distracted by the immediate need for some special case. Having spent this effort in constructing your tools, you'll feel justified to continue with thorough documentation that could prove invaluable later. If you follow these guidelines, you'll create an intellectual tool kit that's easy, effective, and safe to use in a wide variety of applications.

Many tasks of the tool kit you'll construct are best accomplished by turning them into problems about systems of linear equations, usually nonsquare. You should become adept with the elimination techniques described in appendix C for determining whether a system has a solution, for computing a unique solution, and for expressing infinitely many solutions in terms of parameters. Use them in the tool kit.

Standard texts often use vector cross products for some of these tasks. Cross products aren't covered in appendix C. Instead, you should seek methods based on the elimination techniques just mentioned. Cross products are described in detail in section 5.6, and in exercise 5.11.33 you'll be invited to use them to streamline your tools.

Exercise 1. Describe in detail coordinate geometry methods for the following tasks. For each task specify the proper form for the input and for the output, and present an *algorithm*—a computational procedure for hand calculation.

¹⁸ For example, Thomas and Finney [1951] 1979, chapter 11.

Solution. Here are suggested algorithms; you can provide examples.

1. Use the distance formula $PQ = \|P - Q\|$.
2. Use parametric equations $X = Q + (P - Q)t$ for the points X on $P\vec{Q}$.
3. Use X as in (2): $XQ = \|X - Q\| = \|(P - Q)t\| = \|P - Q\| |t| = (PQ)|t| = (PQ)t$.
4. Points P, Q, X are collinear just in case $P = Q$ or $X = Q + (P - Q)t$ for some scalar t . If $P \neq Q$, rearrange this as a 3×1 linear system. To determine whether it has a solution t , find i with $p_i \neq q_i$ then set $t = (x_i - q_i)/(p_i - q_i)$. The points are collinear just in case $x_j = q_j + (p_j - q_j)t$ even when $j \neq i$.
5. Given distinct collinear P, Q, X , find t as in (4). Then P, Q , or X is between the others when $t > 1$, $t < 0$, or $0 < t < 1$.
6. You need $X - P = R - Q$, so $X = P - Q + R$.
7. Points P, Q, R, X are coplanar just in case $\det [P' \ Q' \ R' \ X'] = 0$, as in (7).
8. Solution (7) is a linear equation for the points X on plane PQR . ♦

Exercise 2. Continue exercise 1 for the following tasks. Assume that algorithms are available for all tasks in exercise 1.

<i>Task: given</i>	<i>determine or find</i>
9. lines g and h	whether $g = h$;
10. lines $g \neq h$	whether they're intersecting, parallel, or noncoplanar;
11. intersecting lines $g \neq h$	the point $g \cap h$;
12. point P not on line g	the parallel to g through P ;
13. point P not on line g	the plane through P and g ;
14. intersecting or parallel lines $g \neq h$	their common plane.

Solution. Here are suggested algorithms; you can provide examples.

9. Use the parametric equations to find distinct points P, Q on g and X, Y on h . Use (4) to determine whether P, Q, X and P, Q, Y are collinear triples.
10. Find distinct points P, Q on g and X, Y on h . The lines intersect just in case there exist scalars t and u such that $Q + (P - Q)t = Y + (X - Y)u$, i.e. $(P - Q)t + (Y - X)u = Y - Q$. Arrange this as a 3×2 linear system. If it has a solution, then g and h intersect. If not, use (7) to determine whether P, Q, X, Y are coplanar. If so, then g and h are parallel.
11. Continue the algorithm in (10) to find the solution to the linear system.
12. If g has parametric equations $X = Q + Rt$, then line $X = P + Rt$ is the parallel to g through P .

13. Find distinct points Q, R on g , then use (8) to find a linear equation for the plane PQR .
14. If g and h intersect, use (11) to find $P = g \cap h$. Find points $Q, R \neq P$ on g and h . If $g \parallel h$, find distinct points P, Q on g and any point R on h . Use (8) to find a linear equation for the plane PQR . ♦

Exercise 3. Continue exercises 1 and 2 for the following tasks. Assume that algorithms are available for all previous tasks.

- | | |
|--|---|
| <i>Task: given</i> | <i>determine or find</i> |
| 15. planes δ and ε | whether $\delta = \varepsilon$; |
| 16. planes $\delta \neq \varepsilon$ | whether they intersect; |
| 17. intersecting planes $\delta \neq \varepsilon$ | the line $\delta \cap \varepsilon$; |
| 18. a line and a plane | whether they're incident,
intersecting, or parallel; |
| 19. line g intersecting
plane ε | point $g \cap \varepsilon$; |
| 20. point P not on plane ε | the plane $\delta \parallel \varepsilon$ through P ; |
| 21. line g parallel to plane ε | the plane $\delta \parallel \varepsilon$ through g . |

Solution. Here are suggested algorithms; you can provide examples.

15. Suppose δ and ε have linear equations $S'X = a$ and $T'X = b$. Form columns S' and T' of length four from S and T by appending entries $-a$ and $-b$. The planes are equal just in case one of S' and T' is a scalar multiple of the other. Use the technique of (4) to determine that.
16. The linear equations of distinct planes as in (15) form a 2×3 linear system. The planes intersect if the system has a solution.
17. Continue (16) to find all solutions, expressed in terms of a parameter. The expression constitutes parametric equations for the intersection line.
18. Points X common to g and ε satisfy their parametric and linear equations $X = P + Qt$ and $S'X = a$. That is, $S'P + S'Qt = a$. Arrange this as a linear equation for t . If it has no solution, then $g \parallel \varepsilon$. If it has one, then g and ε are intersecting. If it has infinitely many, they're incident.
19. Continue (18) to compute the solution.
20. If ε has linear equation $S'X = b$, then δ has equation $S'X = a$ for some scalar a . Clearly, $a = S'P$.
21. Find a point P on g and apply (20). ♦

Exercise 4. Continue exercises 1 to 3 for the following tasks. Assume that algorithms are available for all previous tasks.

22. intersecting lines $g \neq h$ whether $g \perp h$;
23. point P not on line g the line $k \perp g$ through P ;
24. point P on line g in the line $k \perp g$ through P
plane ε in ε ;
25. an intersecting line and
plane whether they're $\perp$;
26. point P and line g the plane $\varepsilon \perp g$ through P ;
27. point P and plane ε the line $k \perp \varepsilon$ through P ;
28. planes $\delta \neq \varepsilon$ whether they're $\perp$;
29. line and plane g, δ
with $g \not\subset \delta$ the plane $\varepsilon \perp \delta$ through g ;
30. intersecting lines $g \neq h$ the line $k \perp g, h$;
31. noncoplanar lines g, h the line $k \perp g, h$.

22. Intersecting lines with parametric equations $X = P + Rt$ and $X = Q + St$ are perpendicular just in case $R \perp S$, i.e. $R'S = 0$.
23. Suppose g has parametric equations $X = Q + Rt$. Use (13) to find a linear equation $U'X = a$ for the plane through P and g . If $X = g \cap k$ then $U'X = a$ and $X - P \perp R$, i.e. $R'X = R'P$. Arrange the last two equations as a 2×3 linear system. Solve it for X , then find parametric equations for $k = P\bar{X}$.
24. Suppose g and k have parametric equations $X = Q + Rt$ and $X = Q + St$ and ε has linear equation $U'X = a$. Then $S \perp R, U$, so $R'S = 0$ and $U'S = 0$. Arrange these two equations as a 2×3 linear system and solve it for S .
25. If line $X = P + Qt$ intersects plane $U'X = a$, then they're perpendicular just in case Q is a scalar multiple of U . Use the technique of (4) to determine whether that's so.
26. If g has parametric equations $X = Q + Rt$, then ε has linear equation $R'X = a$ for some scalar a . Clearly, $a = R'P$.
27. If ε has linear equation $R'X = a$, then k has parametric equations $X = P + Rt$.
28. Planes with equations $R'X = a$ and $S'X = b$ are perpendicular just in case $R'S = 0$.
29. Find a point P on g but not on δ . Use (27) to find the line $k \perp \delta$ through P and (19) to find $Q = g \cap \delta$ and $R = k \cap \delta$. Use (8) to find $\varepsilon = PQR$.
30. Use (11) to find $Q = g \cap h$. Find points $P, R \neq Q$ on g, h . Points X on k must satisfy $(P - Q)'(X - Q) = 0 = (R - Q)'(X - Q)$. Arrange these equations as a 2×3 linear system, which you can solve for X . Express the solution in terms of a parameter to get parametric equations for k .

31. Suppose g and h have parametric equations $X = P + Rt$ and $X = Q + St$. Then g intersects h at $X = P + Rt$ and $Y = Q + Su$ for some scalars t and u . Since $X - Y \perp R, S$ the scalars must satisfy equations $R'(Rt - Su + P - Q) = 0$ and $S'(Rt - Su + P - Q) = 0$. Arrange them as a 2×2 linear system and solve it for t and u . ♦

Exercise 5. Find coordinate formulas for the distance between

- | | |
|-------------------------|-------------------------------|
| 1. a point and a line, | 4. a parallel line and plane, |
| 2. two parallel lines, | 5. two parallel planes, |
| 3. a point and a plane, | 6. two noncoplanar lines. |

Solution, Part 1. Consider point P and line g with parametric equations $X = Q + Rt$. For the moment, assume P is not on g . Let X be the foot of the perpendicular to g through P . Since $R \perp X - P$, the corresponding t value satisfies the equation $0 = R'(X - P) = R'(Q + Rt - P)$, hence $t = R'(P - Q)/(R'R)$. The distance from P to g is then $\|X - P\| = \|Q + Rt - P\|$ for this value of t . This formula also yields the appropriate value zero when P is on g .

Part 2. The distance between g and a line $h \parallel g$ is given by the part 1 formula: h has parametric equations $X = P + Rt$ for some P , and the distance is that from P to g .

Part 3. Next, suppose plane ε has linear equation $S'X = a$. Let X be the foot of the perpendicular to ε through P . Then $X - P = St$ for some scalar t , hence $a = S'(P + St) = S'P + S'St$, and $t = (a - S'P)/(S'S) = (a - S'P)/\|S\|^2$. The distance from P to ε is then $\|X - P\| = \|St\| = \|S\| |t| = |a - S'P|/\|S\|$.

Part 4. The distance between ε and a line $g \parallel \varepsilon$ with parametric equations $X = P + Rt$ is given by the part 3 formula.

Part 5. If plane $\delta \parallel \varepsilon$, then δ has linear equation $S'X = b$ for some scalar b . The line with parametric equations $X = St$ intersects ε and δ at points St and Su such that $S'St = a$ and $S'Su = b$, hence $t = a/(S'S) = a/\|S\|^2$ and $u = b/\|S\|^2$. The distance between δ and ε is $\|St - Su\| = \|S(t - u)\| = \|S\| |t - u| = |a - b|/\|S\|$.

Part 6. Exercise 4, task (31) yields formulas for the intersections X and Y of two noncoplanar lines with their common perpendicular. The distance between the lines is $\|X - Y\|$. ♦

Exercise 6, Part 1. How do you find a linear equation for a plane ε if you're given a system of linear parametric equations?

Part 2. How do you find such a system if you're given a linear equation for ε ?

Solution, Part 1. Given linear parametric equations $X = P + Qt + Ru$ for a plane ε , first arrange the equations $Q'S = 0$ and $R'S = 0$ as a

2×3 linear system and solve it for S . Then $S'X = S'P$ is a linear equation for ε .

Part 2. Given a linear equation $S'X = a$ for a plane ε , suppose $s_1 \neq 0$. (Cases $s_2 \neq 0$ and $s_3 \neq 0$ are handled similarly.) The equation reads $s_1x_1 + s_2x_2 + s_3x_3 = 0$. Regard this as a 1×3 linear system. Its solutions, expressed parametrically, are

$$\begin{cases} x_1 = a - \frac{s_2}{s_1}t - \frac{s_3}{s_1}u \\ x_2 = t \\ x_3 = u \end{cases} \quad \text{i.e.} \quad X = \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{s_2}{s_1} \\ 1 \\ 0 \end{bmatrix}t + \begin{bmatrix} -\frac{s_3}{s_1} \\ 0 \\ 1 \end{bmatrix}u. \quad \blacklozenge$$

Exercise 7. Extract from exercises 1 to 6 the material that applies to plane geometry, and present it using two-dimensional analytic formulas.

Solution. Here are the appropriate tasks. You can adapt the formulas derived in earlier exercises.

Task:

- | <i>given</i> | <i>determine or find</i> |
|---|--|
| 1. points P and Q , ... | the distance PQ ; |
| 2. points $P \neq Q$, ... | the line $\vec{PQ}$; |
| 3. points $P \neq Q$ and a scalar $r > 0$, . . . | point R on $\vec{PQ}$ with $PR/PQ = r$; |
| 4. three points, ... | whether they're collinear; |
| 5. three distinct collinear points, ... | which one lies between the others; |
| 6. noncollinear points P, Q, R , ... | point X of parallelogram $PQRX$; |
| 7. lines g and h , ... | whether $g = h$; |
| 8. lines $g \neq h$, ... | whether they intersect; |
| 9. intersecting lines $g \neq h$, ... | the point $g \cap h$; |
| 10. point P not on line g , ... | the line parallel to g through P ; |
| 11. intersecting lines $g \neq h$, ... | whether $g \perp h$; |
| 12. point P and line g , ... | the line $k \perp g$ through P ; |
| 13. a point and a line, ... | the distance between them; |
| 14. parallel lines, ... | the distance between them; |
| 15. parametric equations for line g , . . . | a linear equation for g ; |
| 16. a linear equation for line g , ... | parametric equations for g . $\blacklozenge$ |

The last exercise of this chapter investigates an alternate definition for the interior of a tetrahedron. Clearly, a point lies in the interior if it's between, but different from, two points on opposite edges. Is the converse true? The

problem is included here because analytic methods may work better than synthetic ones.

Exercise 8. Select two opposite edges of a tetrahedron T . Does every point interior to T lie between points on these edges?

Solution. Select edges $\overline{PQ}$ and $\overline{RS}$ of tetrahedron $T = PQRS$, and a point X interior to T . Then X lies between P and a point Y interior to $\triangle QRS$, and Y lies between Q and a point Z interior to $\overline{RS}$. There exist scalars x , y , and z between 0 and 1 such that

$$\begin{aligned} X &= P(1 - x) + Yx, \\ Y &= Q(1 - y) + Zy, \\ Z &= R(1 - z) + Sz. \end{aligned}$$

These equations entail

$$\begin{aligned} X &= P(1 - x) + (Q(1 - y) + Zy)x \\ &= P(1 - x) + Q(1 - y)x + Zyx \\ &= P(1 - x) + Q(1 - y)x + (R(1 - z) + Sz)yx \\ &= P(1 - x) + Q(1 - y)x + R(1 - z)yx + Szyx. \end{aligned}$$

Set $a = 1 - xy$, $b = (1 - x)/a$, and $c = 1 - z$, so that a , b , and c lie between 0 and 1. Observe:

$$\begin{aligned} ab &= 1 - x \\ (1 - b)a &= (1 - y)x \\ (1 - a)c &= (1 - z)yx \\ (1 - a)(1 - c) &= xyz \\ X &= Pab + Q(1 - b)a + R(1 - a)c + S(1 - a)(1 - c) \\ &= (Pb + Q(1 - b))a + (Rc + S(1 - c))(1 - a). \end{aligned}$$

Thus X lies between points $Pb + Q(1 - b)$ and $Rc + S(1 - c)$ interior to $\overline{PQ}$ and $\overline{RS}$. ♦

5.11

Solutions to exercises on triangle and circle geometry

Concepts

Analytic proofs of this chapter's major theorems
Circumcenter, incenter, circumsphere, insphere, and centroid of a tetrahedron
Orthocenters of rhombic tetrahedra
Homothetic centers of two circles
Finding triangle measurements from SAS, SSS, ASA, and SSA data
Tangent sum, difference, double angle, and half angle formulas
Chebyshev polynomials
Approximating sine values
Survey of spherical trigonometry
Using cross products and determinants in analytic geometry
Exercises that can utilize algebra software

This section contains thirty-eight exercises related to the earlier parts of the chapter. They're gathered together here because the material covered in chapter 5 is so tightly interrelated. Even routine problems such as exercise 1, intended to provide computational experience with the concepts in section

5.1, benefit from application of formulas derived much later in the chapter. Often a problem that seems to be about one subject yields most easily to methods introduced to study another. You'll find here a few straightforward applications of the major results in the text, and of the methods used in deriving them. But many of these exercises carry on the tradition started in chapter 4; they complete the development of theories and techniques started in the text, or extend them to new subjects.

For example, exercises 7 to 9 introduce the theory of tetrahedra: the three-dimensional analog of this whole chapter. Exercises 21 to 24 complete the development of plane trigonometry begun in chapters 3 and 4 and pursued hotly in section 5.5. With these exercises, the book contains the entire geometric content of the familiar trigonometry course, seen from an advanced viewpoint. Another family of exercises, 28 to 32, carries out a parallel development of spherical trigonometry, a neglected but highly practical branch of mathematics whose repertoire of applications is increasing.

Exercises in section 4.10 encouraged you to assemble an analytic geometry tool kit for use in a wide variety of applications. Analytic techniques were developed and extended in sections 3.11 and 5.6, using vector algebra notation and concepts summarized in appendix C. Several exercises in the present section apply and enhance those techniques. Most of the straightforward computations in exercises 1 and 2 are probably familiar to you from more elementary courses. But these quickly give way to much more complicated examples. Analytic proofs of some of the major theorems of triangle geometry require algebraic computations that touch or exceed the limit of human capability. Joint efforts of computer scientists and mathematicians have produced algebra software that can help with such work. You're invited to use it to carry out some of the computations in the last two exercises, which feature an analytic proof of Feuerbach's theorem, the climax of this chapter.

Related exercises are gathered under headings. Under a particular heading, the simpler exercises usually precede the deeper, more complex ones.

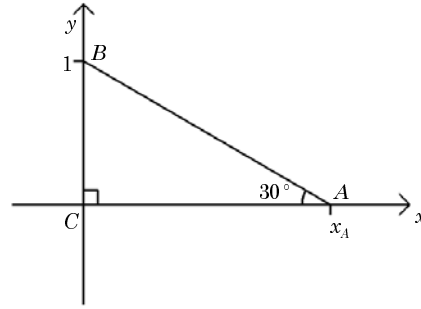
Basic exercises

The first few exercises in this section illustrate its most basic ideas. This arrangement shows why the exercises are gathered at the end of the chapter. An efficient solution of exercise 1, for example, will probably use concepts from sections 5.1, 5.5, and 5.9. Other basic exercises involve material covered in 5.7 and 5.8. Exercise 1 could be stated simply, *illustrate with a particular triangle as many as possible of the concepts introduced in chapter 5.*

Exercise 1. Consider in detail $\triangle ACB$ shown with a coordinate system in figure 5.11.1. Calculate the following scalars and vectors:

Figure 5.11.1

Exercise 1



centroid	G	inradius	r
circumcenter . . .	O	Nagel point	N
circumradius . . .	R	orthocenter	H
incenter	I		

Write all scalars in lowest terms with integer denominators. *Suggestion:* The Nagel point has y coordinate ≈ 0.27 .

Solution. Since edge $a = 1$ and $m\angle A = 30^\circ$, the hypotenuse $c = 2a = 2$. Edge $b = \sqrt{3}$ by Pythagoras' theorem, so $A = \langle \sqrt{3}, 0 \rangle$. The semiperimeter is $s = \frac{1}{2}(a + b + c) = \frac{1}{2}(3 + \sqrt{3})$. Since the circumcenter of a right triangle is the midpoint of its hypotenuse, $O = C' = \frac{1}{2}(A + B) = \langle \frac{1}{2}\sqrt{3}, \frac{1}{2} \rangle$ and $R = \frac{1}{2}c = 1$. The orthocenter of a right triangle is the vertex of its right angle, so $H = C$. The centroid is $\frac{2}{3}$ of the way from C to C' , so $G = \frac{2}{3}C' = \frac{2}{3}O$. The area $\Delta = \frac{1}{2}ab = \frac{1}{2}\sqrt{3}$. Equation $\Delta = rs$ yields $r = \Delta/s = \frac{1}{2}(-1 + \sqrt{3})$. The incenter $I = \langle r, r \rangle$. Equations $\Delta = r_a(s - a) = r_b(s - b)$ yield $r_a = \Delta/(s - a) = \frac{1}{2}(3 - \sqrt{3})$ and $r_b = \Delta/(s - b) = \frac{1}{2}(1 + \sqrt{3})$. The excircles opposite A and B touch the edge lines at $E_{aa} = \langle 0, r_a \rangle$ and $E_{bb} = \langle r_b, 0 \rangle$. $\vec{AE}_{aa}$ and $\vec{BE}_{bb}$ have linear equations

$$\frac{y - 0}{x - b} = \frac{r_a - 0}{0 - b} \quad \frac{y - a}{x - 0} = \frac{0 - a}{r_b - 0}$$

$$y = -\frac{r_a}{b}x + r_a \quad y = -\frac{a}{r_b}x + a.$$

Thus you can find their intersection, the Nagel point N , by solving

$$-\frac{r_a}{b}x + r_a = -\frac{a}{r_b}x + a.$$

That yields $x = 1$, and from either of the preceding equations you'll get $y = 2 - \sqrt{3}$. Thus, $N = \langle 1, 2 - \sqrt{3} \rangle \approx \langle 1.00, 0.27 \rangle$. In summary,

centroid	G	$\langle \frac{1}{3}\sqrt{3}, \frac{1}{3} \rangle$
circumcenter	O	$\langle \frac{1}{2}\sqrt{3}, \frac{1}{2} \rangle$

circumradius	R	1
incenter	I	$\langle \frac{1}{2}(-1 + \sqrt{3}), \frac{1}{2}(-1 + \sqrt{3}) \rangle$
inradius	r	$\frac{1}{2}(-1 + \sqrt{3})$
Nagel point	N	$\langle 1, 2 - \sqrt{3} \rangle$
orthocenter	H	$\langle 0, 0 \rangle$ ♦

Exercise 1 used a particularly simple triangle to limit computational complexity. That restricts somewhat the repertoire of concepts it can illustrate. You may want to pose and solve similar exercises with less special triangles, and calculate more features. For example, try an isosceles triangle that's not right, and show that the feet of its altitudes do all lie on its Feuerbach circle.

The next exercise uses the same analytic methods in a general context. It's really three related exercises for the analytic geometry tool kit that you assembled in section 4.10. For any triangle $\triangle ABC$ there's a Cartesian coordinate system with the arrangement shown in figure 5.11.2.

Exercise 2. Prove the edge bisectors, altitudes, and medians theorems analytically, using $\triangle ABC$ in figure 5.11.2.

Solution. The edge midpoints are

$$\begin{aligned} A' &= \langle \frac{1}{2} x_B, \frac{1}{2} y_C \rangle \\ B' &= \langle \frac{1}{2} x_A, \frac{1}{2} y_C \rangle \\ C' &= \langle \frac{1}{2} (x_A + x_B), 0 \rangle \end{aligned}$$

and the perpendicular bisectors j , k , and l of the corresponding edges have x, y equations

$$\begin{aligned} j: \quad & (x - x_B)^2 + y^2 = x^2 + (y - y_C)^2 \\ & 2y_C y - 2x_B x = y_C^2 - x_B^2 \\ k: \quad & 2y_C y - 2x_A x = y_C^2 - x_A^2 \\ l: \quad & x = \frac{1}{2} (x_A + x_B). \end{aligned}$$

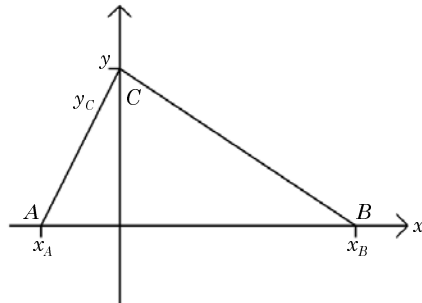


Figure 5.11.2
Exercise 2

The intersection $O = \langle x, y \rangle$ of j and k — the circumcenter, not the origin — satisfies the first two equations, hence their difference $2(x_A - x_B)x = x_A^2 - x_B^2$, which is equivalent to $x = \frac{1}{2}(x_A + x_B)$. Therefore, O lies on l : the edge bisectors theorem.

The edges opposite A and B have slopes $-y_C/x_B$ and $-y_C/x_A$, hence the altitude lines h_A and h_B through A and B have x, y equations

$$h_A: y = (x_B/y_C)(x - x_A) \quad h_B: y = (x_A/y_C)(x - x_B).$$

Their intersection $H = \langle x, y \rangle$ satisfies the equation $(x_B/y_C)(x - x_A) = (x_A/y_C)(x - x_B)$, hence $x(x_B - x_A)/y_C = 0$, so $x = 0$. Therefore, H lies on the altitude line through C : the altitudes theorem.

The medians m_a , m_b , and m_c through A , B , and C satisfy equations

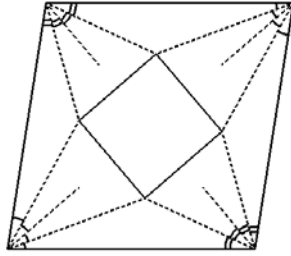
$$\begin{aligned} m_a: \quad & \frac{x - x_A}{y - 0} = \frac{\frac{1}{2}x_B - x_A}{\frac{1}{2}y_C - 0} \\ & \frac{x - x_A}{y} = \frac{x_B - 2x_A}{y_C} \\ & y_C x + (2x_A - x_B)y - x_A y_C = 0 \\ m_b: \quad & y_C x + (2x_B - x_A)y - x_B y_C = 0 \\ m_c: \quad & \frac{x - 0}{y - y_C} = \frac{\frac{x_A + x_B}{2} - 0}{0 - y_C} \\ & \frac{x}{y - y_C} = -\frac{x_A + x_B}{2y_C} \\ & 2y_C x + (x_A + x_B)y - (x_A + x_B)y_C = 0. \end{aligned}$$

By theorem 5.6.12, these lines have a common point because they're not parallel and

$$\det \begin{bmatrix} y_C & 2x_A - x_B & -x_A \\ y_C & 2x_B - x_A & -x_B \\ 2y_C & x_A + x_B & -(x_A + x_B) \end{bmatrix} = 0.$$

(The third row is the sum of the first two.) That's the medians theorem. ♦

The angle bisectors theorem doesn't yield easily to these methods, mainly because the formulas for the slopes of the bisectors are too complicated. You'll find an exercise on that theorem at the end of this section, with a suggestion to use mathematical software to help with the algebra.

**Figure 5.11.3**

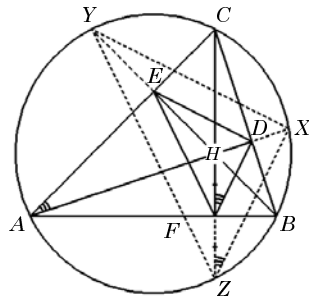
Exercise 3

Exercise 3. Prove that the quadrilateral in figure 5.11.3, formed from the intersections of adjacent quadrisectioners of the angles of a rhombus, is a square.

Solution. The diagonals of a rhombus meet at its center O . By theorems 3.7.18, 3.7.19, and the SAS congruence theorem, they divide it into four triangles $\mathcal{T}_1$ to $\mathcal{T}_4$ with right angles at O . Any two adjacent quadrisectioners bisect the acute angles of one of the $\mathcal{T}_k$, hence meet at its incenter I_k . This point also lies on one of the bisectors g and h of the right angles at O ; notice that $g \perp h$. Use the congruence of triangles $\mathcal{T}_k$ and the ASA congruence theorem to show that the four segments $\overline{OI_k}$ are congruent. The diagonals of the quadrilateral of incenters are thus congruent, perpendicular and bisect each other. Use the SAS theorem to show that it's square.¹ ♦

The next exercise has a particularly nice figure. Draw it!

Exercise 4. The altitude lines of an oblique $\triangle ABC$ through vertices A , B , and C meet its circumcircle at points $X \neq A$, $Y \neq B$, and $Z \neq C$. Show that the edge lines of $\triangle XYZ$ are parallel to those of the orthic triangle, and these triangles are similar, with ratio 2.

**Figure 5.11S.1**

Exercise 4 solution

¹ Compare the solution listed in the bibliography as *Mathematics magazine* 1989.

Also, $AA' < AB' + B'A'$; that is,

$$m_a < \frac{1}{2}b + \frac{1}{2}c.$$

Similarly,

$$\begin{aligned} m_b &< \frac{1}{2}c + \frac{1}{2}a \\ m_c &< \frac{1}{2}a + \frac{1}{2}b. \end{aligned}$$

Adding these three inequalities, you get

$$m_a + m_b + m_c < p. \blacklozenge$$

Exercises on tetrahedra

Most of the plane geometric results covered in this chapter have three-dimensional analogs. You can derive some of them, such as the three in exercise 7, exactly as in plane geometry. Understanding the three-dimensional situations strengthens your hold on the basic plane theory. In other cases, such as exercise 8, the analogy isn't perfect; you must adjust the statement to get an interesting result. These two exercises will seem somewhat more difficult than the basic ones included under the previous heading, but that probably stems only from the effort necessary to visualize unfamiliar three-dimensional configurations. Exercise 9, the last on tetrahedra, is considerably more tedious. Its solution uses methods you've encountered already in the proof of theorem 5.9.2 and in exercise 4.6.6.

Exercise 7. Invent and prove three-dimensional analogs of the edge bisectors, angle bisectors, and medians theorems and related results in section 5.1. Use tetrahedra and spheres in place of triangles and circles. Relate the centroid of a tetrahedron to its vertices, the midpoints of its edges, and the centroids of its faces.

Solution. The perpendicular bisectors and angle bisectors theorems have straightforward analogs; bisect the edges and dihedral angles with planes. The bisectors meet at the *circumcenter* and *incenter* of the tetrahedron: the centers of its *circumscribed* and *inscribed* spheres. The *medians* join each vertex to the centroid of the opposite face. They all pass through a point—the *centroid* of the tetrahedron— $\frac{3}{4}$ of the way from each vertex to the opposite face. The centroid is also the common midpoint of the three segments joining midpoints of opposite edges. $\blacklozenge$

Exercise 8. This exercise develops a three-dimensional altitudes theorem. Since the altitude lines of a tetrahedron don't necessarily have a common point, the problem is to describe when that happens. First, you need some preliminary results relating tetrahedra and parallelepipeds.

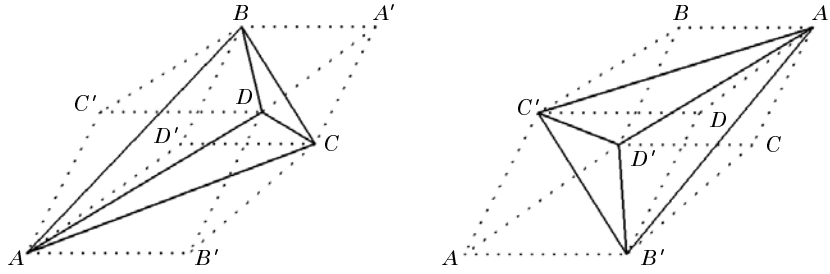


Figure 5.11.4 Twin tetrahedra

Part 1. From a parallelepiped $\mathcal{P}$ with bases $AB'CD'$ and $C'DA'B$ and directrix $\vec{AC'}$, construct *twin* tetrahedra $ABCD$ and $A'B'C'D'$ as suggested by figure 5.11.4. Show that they're congruent, that $BCD \parallel B'C'D'$, and that every tetrahedron $ABCD$ can be constructed this way from some parallelepiped.

Part 2. $ABCD$ is called *rhombic* if the faces of $\mathcal{P}$ are all rhombic. Show that $ABCD$ is rhombic if and only if each edge lies in a plane perpendicular to the opposite edge line.

Part 3. Show that the altitude lines of a rhombic tetrahedron intersect at its twin's circumcenter.

Part 4. Show that if any two pairs of altitude lines of a tetrahedron intersect, then it's rhombic.

Part 5. The intersection of the four altitude lines of a rhombic tetrahedron is called its *orthocenter*. Show that the common perpendiculars of the three pairs of opposite edges of a rhombic tetrahedron intersect at its orthocenter.

*Solution, Parts 1 and 2.*² These are simple, except for the proof that $BCD \parallel B'C'D'$, which is most easily done with analytic geometry.

Part 3. Let $ABCD$ be a rhombic tetrahedron with twin $A'B'C'D'$, and h be the perpendicular to plane $B'C'D'$ through its circumcenter. The twin's circumcenter H must lie on h , and so must A because $AB' = AC' = AD'$. By part (1), $h \perp BCD$, so h is the altitude line through A . Similarly, H must lie on the other altitude lines.

Part 4. For each vertex V , let F_V denote the intersection of the opposite face with the altitude line through V . Suppose $\vec{AF}_A$ and $\vec{BF}_B$ meet at a point H . Then $HAB \perp BCD, CDA$, so $HAB \perp \vec{CD} = BCD \cap CDA$. Thus $\vec{AB}$ is parallel to a line perpendicular to $\vec{CD}$. These are diagonal lines of opposite faces of the parallelepiped $\mathcal{P}$ from which $ABCD$ is constructed,

² A source for this kind of material is Altshiller-Court's classic text [1935] 1964, chapter 4.

hence those faces are rhombic. If another pair of altitude lines meet, then another pair of opposite faces of $\mathcal{T}$ are rhombic, so all its faces are.

Part 5. $\vec{BF}_A$ lies in BCD and HAB , and $\vec{AF}_B$ lies in CDA and HAB . These lines are coplanar but can't be parallel, so they must meet at a point P in $\vec{CD} = BCD \cap CDA$. Then H is the orthocenter of ΔPAB because $\vec{AF}_A \perp \vec{BP}$ and $\vec{BF}_B \perp \vec{AP}$, hence $\vec{PH} \perp \vec{AB}$. That is, H lies on the common perpendicular $\vec{PH}$ of $\vec{AB}$ and $\vec{CD}$. ♦

Exercise 9. Demonstrate the equivalence of the following three properties of a tetrahedron $\mathcal{T}$:

- (a) The incircles of the faces of $\mathcal{T}$ are tangent in pairs.
- (b) There's a sphere tangent to all edges of $\mathcal{T}$.
- (c) The sums of the lengths of the three pairs of opposite sides of $\mathcal{T}$ are equal.

Solution. Let Γ_1 to Γ_4 and I_1 to I_4 denote the incircles and incenters of the faces opposite vertices V_1 to V_4 .

Assume (a), so that each Γ_i and Γ_j touch at a point T_{ij} . Since $\vec{I}_i T_{ij}$, $\vec{I}_j T_{ij} \perp \vec{V}_k V_l$ when i, j, k , and l are distinct, $\vec{I}_i T_{ij} \perp \vec{V}_k V_l$ and the perpendiculars g_i and g_j through I_i and I_j to planes $V_i V_k V_l$ and $V_j V_k V_l$ lie in plane $\vec{I}_i T_{ij} \vec{I}_j$. Therefore g and h intersect at a point O , the center of a sphere Σ tangent to all edge lines except possibly $\vec{V}_i V_j$. Repeat the argument with a different pair i, j to show that Σ and $\vec{V}_i V_j$ are also tangent. Thus (a) implies (b).

Condition (b) obviously implies (a). The next step is to show that (a) implies (c). Assume (a). Then

$$\begin{aligned} V_1 T_{34} &= T_{23} V_1 & T_{34} V_2 &= V_2 T_{14} \\ V_3 T_{12} &= T_{14} V_3 & T_{12} V_4 &= V_4 T_{23} \\ V_1 V_2 + V_3 V_4 &= V_1 T_{34} + T_{34} V_2 + V_3 T_{12} + T_{12} V_4 \\ &= V_2 T_{14} + T_{14} V_3 + V_4 T_{23} + T_{23} V_1 \\ &= V_2 V_3 + V_4 V_1. \end{aligned}$$

Similarly, this sum equals $V_1 V_3 + V_2 V_4$. Thus, (a) implies (c).

The last step is to show that (c) implies (a). Suppose Γ_i touches $\vec{V}_j V_k$ at point P_{ijk} and assign directions to each of the edge lines so that $(V_1 \text{ to } V_2)$, $(V_2 \text{ to } V_3)$, $(V_3 \text{ to } V_4)$, and $(V_4 \text{ to } V_1)$ are all positive. Now assume (c), so that

$$\begin{aligned} &(V_1 \text{ to } P_{312}) + (P_{312} \text{ to } P_{412}) + (P_{412} \text{ to } V_2) \\ &\quad + (V_3 \text{ to } P_{134}) + (P_{134} \text{ to } P_{234}) + (P_{234} \text{ to } V_4) \\ &= (V_1 \text{ to } V_2) + (V_3 \text{ to } V_4) \\ &= (V_2 \text{ to } V_3) + (V_4 \text{ to } V_1) \end{aligned}$$

$$= (V_2 \text{ to } P_{123}) + (P_{123} \text{ to } P_{423}) + (P_{423} + V_3) \\ + (V_4 \text{ to } P_{341}) + (P_{341} \text{ to } P_{241}) + (P_{241} \text{ to } V_1).$$

Since $(V_1 \text{ to } P_{312}) = (P_{341} + P_{241}) + (P_{241} \text{ to } V_1)$,

$$(P_{312} \text{ to } P_{412}) + (P_{412} \text{ to } V_2) + (V_3 \text{ to } P_{134}) \\ + (P_{134} \text{ to } P_{234}) + (P_{234} \text{ to } V_4) \\ = (V_2 \text{ to } P_{123}) + (P_{123} \text{ to } P_{423}) + (P_{423} \text{ to } V_3) + (V_4 \text{ to } P_{341}).$$

Moreover, $(V_3 \text{ to } P_{134}) = (P_{123} \text{ to } P_{423}) + (P_{423} \text{ to } V_3) = (P_{123} \text{ to } V_3)$, so

$$(P_{312} \text{ to } P_{412}) + (P_{412} \text{ to } V_2) + (P_{134} \text{ to } P_{234}) + (P_{234} \text{ to } V_4) \\ = (V_2 \text{ to } P_{123}) + (V_4 \text{ to } P_{341}), \\ (P_{312} \text{ to } V_2) + (P_{134} \text{ to } V_4) = (V_2 \text{ to } P_{123}) + (V_4 \text{ to } P_{341}).$$

But

$$(P_{312} \text{ to } V_2) = (P_{342} \text{ to } V_2) \quad (P_{134} \text{ to } V_4) = (V_4 \text{ to } P_{142}) \\ (V_2 \text{ to } P_{123}) = (P_{142} \text{ to } V_2) \quad (V_4 \text{ to } P_{341}) = (V_4 \text{ to } P_{342}),$$

so

$$(P_{342} \text{ to } V_2) + (V_4 \text{ to } P_{142}) = (P_{142} \text{ to } V_2) + (V_4 \text{ to } P_{342}) \\ (P_{342} \text{ to } V_2) + (V_4 \text{ to } P_{142}) + (V_2 \text{ to } P_{142}) + (P_{342} \text{ to } V_4) = 0 \\ 2(P_{342} \text{ to } P_{142}) = 0.$$

Therefore $P_{342} = P_{142}$ —that is, I_1 and I_3 touch $V_2 \vec{V}_4$ at the same point. You can verify the other common tangencies the same way. Therefore, (c) implies (a). ♦

Exercises on Menelaus', Desargues', Ceva's, and Feuerbach's theorems

Exercises under this heading include applications of major theorems of this section, additional uses of methods employed to derive them, and an alternate proof. Analytic proofs are included under a later heading. Exercise 10 applies Menelaus' theorem to quadrilaterals. Can you generalize it to handle polygons with more than four vertices?

Exercise 10. Assign scales to the lines in figure 5.11.5, in order to use directed distances. Prove that

$$\frac{A \text{ to } E'}{E' \text{ to } C'} \frac{C' \text{ to } D'}{D' \text{ to } B} \frac{B \text{ to } D}{D \text{ to } C} \frac{C \text{ to } E}{E \text{ to } A} = 1.$$

Solution. Let $X = \vec{AB} \cap \vec{DE}$ and assign a scale to $\vec{AB}$. Then

$$\frac{A \text{ to } E'}{E' \text{ to } C'} \frac{C' \text{ to } D'}{D' \text{ to } B} \frac{B \text{ to } D}{D \text{ to } C} \frac{C \text{ to } E}{E \text{ to } A} \\ = \frac{A \text{ to } E'}{E' \text{ to } C'} \frac{C' \text{ to } D'}{D' \text{ to } B} \frac{B \text{ to } X}{X \text{ to } A} \cdot \frac{A \text{ to } X}{X \text{ to } B} \frac{B \text{ to } D}{D \text{ to } C} \frac{C \text{ to } E}{E \text{ to } A} = (-1)(-1) = 1. \quad \blacklozenge$$

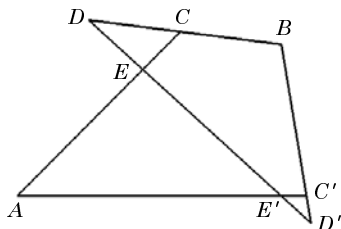


Figure 5.11.5

Exercise 10

Exercise 11. An intersection of two distinct common tangents to two distinct circles is called their *external homothetic center* if the circles lie on the same side of both tangents, and their *internal homothetic center* if they're on different sides of both tangents. Consider three circles, each two of which have an external homothetic center. Show that those three homothetic centers are collinear. What about internal homothetic centers? Combinations of internal and external homothetic centers?

Solution. Use this notation:

$\Gamma_A, \Gamma_B, \Gamma_C \dots$ circles with centers A, B, C ;
 $r_A, r_B, r_C \dots$ their radii;
 $X \dots \dots \dots$ external homothetic center of Γ_B and Γ_C ;
 $Y \dots \dots \dots$ external homothetic center of Γ_C and Γ_A ;
 $Z \dots \dots \dots$ external homothetic center of Γ_A and Γ_B .

Then X, Y , and Z lie on $a = \vec{BC}$, $b = \vec{CA}$, and $c = \vec{AB}$. If these are the same line, the result holds. If not, assign scales to them in order to use directed distances, and apply Menelaus' theorem to $\triangle ABC$:

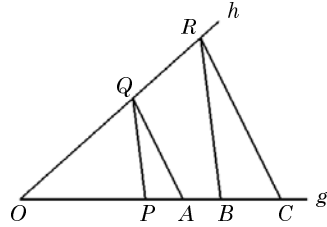
$$\frac{A \text{ to } Z}{Z \text{ to } B} \frac{B \text{ to } X}{X \text{ to } C} \frac{C \text{ to } Y}{Y \text{ to } A} = \left(-\frac{r_A}{r_B} \right) \left(-\frac{r_B}{r_C} \right) \left(-\frac{r_C}{r_A} \right) = -1.$$

That implies collinearity of X, Y , and Z .

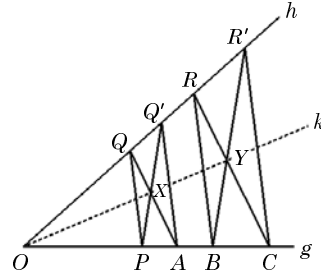
If two pairs of circles have internal homothetic centers and the third has an external center, then those centers are collinear by the same argument; you get two minus signs and one plus. But other combinations of homothetic centers won't generally fall on a common line.³ ♦

Exercise 12. Consider points $D, E, F \neq A, B, C$ on the edge lines opposite these vertices of $\triangle ABC$. Suppose lines $\vec{AD}$, $\vec{BE}$, and $\vec{CF}$ are copencilar; let $\vec{BC}$ meet $\vec{EF}$ at A^* , $\vec{CA}$ meet $\vec{FD}$ at B^* , and $\vec{AB}$ meet $\vec{DE}$ at C^* . Show that A^*, B^* , and C^* are collinear.

³ McLeary 1982 and Heil 1984 contain solutions based on Desargues' theorem, and on a three-dimensional construction analogous to Whitehead's proof of Desargues' theorem.

**Figure 5.11.6**

$(O \text{ to } A)(O \text{ to } B) = (O \text{ to } C)$
if $(O \text{ to } P) = 1$

**Figure 5.11S.3**

Exercise 14
solution, part 2

Solution. Apply Desargues' theorem to $\triangle ABC$ and $\triangle DEF$. ♦

Exercise 13. Draw the scene in figure 1.1.3 on a vertical panel placed obliquely, so that no tile edge is parallel or perpendicular to the image plane. Start with a perspective image of a single tile, analogous to figure 1.1.6. Then construct the image of the rest of the tiling. Imitate the section 1.1 discussion, using Desargues' theorem to show that your technique is consistent.

Solution. Repeat that discussion almost verbatim, using the concurrent case of Desargues' theorem in figure 5.3.1 instead of the parallel case. For point O , use the intersection of edge lines $P\vec{Q}$ and $R\vec{S}$ in the new image. ♦

Exercise 14. Figure 5.11.6 displays a geometric algorithm for multiplying coordinates of points on a line g , relative to a particular scale. Let O and P be the points on g with coordinates 0 and 1, and select any line $h \neq g$ through O and any point $Q \neq O$ on h . Given any points A and B on g , find R on h and C on g so that $B\vec{R} \parallel P\vec{Q}$ and $R\vec{C} \parallel Q\vec{A}$.

Part 1. Show that

$$(O \text{ to } A)(O \text{ to } B) = (O \text{ to } C).$$

This equation of course implies that you get the same point C regardless of your earlier choices of h and Q .

Part 2. Now suppose you knew Desargues' theorem but not the equation in part (1). Choose a point $Q' \neq Q, O$ on h . Construct C' according to the algorithm, using Q' in place of Q . Show how Desargues' theorem implies $C = C'$.

Part 3. Same as part (2), but now choose a line $h' \neq g, h$ through O , select $Q' \neq O$ on h' , and construct C' according to the algorithm, using Q' and h' in place of Q and h .

Solution, Part 1. $\triangle OAQ \sim \triangle OCR$ and $\triangle OQP \sim \triangle ORB$, so

$$\frac{OA}{OC} = \frac{OQ}{OR} = \frac{OP}{OB}.$$

In figure 5.11.6, where A and B are both on the same side of O as P , this implies $(O \text{ to } A)(O \text{ to } B) = (OA)(OB) = (OP)(OC) = OC = (O \text{ to } C)$. You should check that the directed distances work out properly in the other cases, too.

Part 2. (See Figure 5.11S.3.) Find the intersection X of $\vec{QA}$ and $\vec{PQ}'$. Let $k = \vec{OX}$, and find the intersection Y of k and $\vec{RC}$. Apply Desargues' theorem to $\triangle PQX$ and $\triangle BRY$ to get $\vec{XP} \parallel \vec{YB}$. Thus Y lies on $\vec{BR}'$. Apply Desargues' theorem to $\triangle XAQ'$ and $\triangle YCR'$ to get $\vec{AQ}' \parallel \vec{CR}'$. Thus $\vec{CR}' = \vec{C'R'}$, so $C = C'$.

Part 3. Similar to part (2). ♦

In his 1899 book *Foundations of geometry*, David Hilbert showed how to formulate geometric axioms without mentioning scalars, how to *define* scalars as the points on a particular line g , and how to use Desargues' theorem as in the preceding exercise to define the product of two scalars. He handled addition similarly, and derived all the standard algebraic properties of these operations from his geometric axioms.

Exercise 15 demonstrates another way to prove Ceva's theorem, in addition to the two proofs in section 5.4. In exercise 16, you can apply the same method to derive some related formulas.

Exercise 15. Prove statement (i) of Ceva's theorem by considering ratios of areas of the six triangles that make up $\triangle ABC$ in figure 5.4.1. Can you extend that method to apply to any of statements (ii) to (vi)?

$$\begin{aligned} \text{Solution. } \frac{A \text{ to } Z}{Z \text{ to } B} &= \frac{a \triangle AZW}{a \triangle ZBW} = \frac{a \triangle AZC}{a \triangle ZBC} \\ &= \frac{a \triangle AZW - a \triangle AZC}{a \triangle ZBW - a \triangle ZBC} = \frac{a \triangle AWC}{a \triangle WBC}. \end{aligned}$$

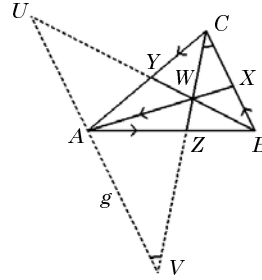
Similarly,

$$\frac{B \text{ to } X}{X \text{ to } C} = \frac{a \triangle BWA}{a \triangle WCA} \quad \frac{C \text{ to } Y}{Y \text{ to } A} = \frac{a \triangle CWB}{a \triangle WAB}.$$

Multiplying the ratios of directed distances, you get a product of ratios of areas. Their numerators and denominators all cancel. ♦

Exercise 16. Prove that in figure 5.4.1,

$$\frac{WX}{AX} + \frac{WY}{BY} + \frac{WZ}{CZ} = 1 \quad \frac{AW}{WX} = \frac{AY}{CY} + \frac{AZ}{ZB}.$$

Figure 5.11S.4Exercise 16 solution,
part 2

Solution. WX/AX is the ratio of the altitudes of $\triangle BWC$ and $\triangle ABC$, hence it's the ratio of their areas. Similar results hold for the other vertices, so

$$\begin{aligned} \frac{WX}{AX} + \frac{WY}{BY} + \frac{WZ}{CZ} &= \frac{a \triangle BWC}{a \triangle ABC} + \frac{a \triangle CWA}{a \triangle ABC} + \frac{a \triangle AWB}{a \triangle ABC} \\ &= \frac{a \triangle ABC}{a \triangle ABC} = 1. \end{aligned}$$

To derive the second equation, construct $g \parallel \vec{BC}$ through A , as in figure 5.11S.4. It intersects $\vec{CZ}$ and $\vec{BY}$ at points U and V . Then $\triangle AUW \sim \triangle XBW$ and $\triangle AVW \sim \triangle XCW$, so

$$\frac{AW}{WX} = \frac{UA}{BX} = \frac{AV}{XC} \quad \frac{AW}{WX} = \frac{UA + AV}{BX + XC} = \frac{UV}{BC} \cdot \blacklozenge$$

Exercise 17 is a converse of part of theorem 5.7.1. To solve it, you may need to analyze when positive scalars x , y , and z have the same geometric and arithmetic means—that is, when $\sqrt[3]{xyz} = \frac{1}{3}(x + y + z)$.

Exercise 17. Show that if $\triangle AZW$, $\triangle BXW$, and $\triangle CYW$ in figure 5.4.1 have equal areas, then W is the centroid of $\triangle ABC$.

*Solution.*⁴ Let $t = a \triangle AZW = a \triangle BXW = a \triangle CYW$, $x = BX/XC$, $y = CY/YA$, and $z = AZ/ZB$, so that $a \triangle ZBW = t/z$, $a \triangle XCW = t/x$, and $a \triangle YAW = t/y$. Then $a \triangle ABX = x(a \triangle CAX)$:

$$2t + \frac{t}{z} = x \left(t + \frac{t}{x} + \frac{t}{y} \right).$$

Multiplying both sides by yz/t , you get $2yz + y = xyz + yz + zx$. By Ceva's theorem, $xyz = 1$, so

$$yz + y = 1 + zx.$$

⁴ See the bibliography entry *Mathematics magazine* 1984.

Similarly,

$$\begin{aligned}zx + z &= 1 + xy \\xy + x &= 1 + yz.\end{aligned}$$

Adding the preceding three equations yields

$$xy + yz + zx + x + y + z = 3 + xy + yz + zx,$$

hence $x + y + z = 3$.

Without loss of generality, you can assume $x \leq y \leq z$. Suppose $x < z$ — this will lead to a contradiction. If $1 \leq x$, then $1 \leq y$ and $1 < z$, and you'd have $3 < x + y + z$. If $z \leq 1$, then $y \leq 1$ and $x < 1$, and you'd have $x + y + z < 3$. Thus $x < 1 < z$, so $x = 1 - h$ and $z = 1 + k$, where $0 < h < 1$ and $0 < k$. Therefore $y = 3 - x - z = 1 + h - k$, so

$$1 = xyz = (1 - h)(1 + h - k)(1 + k).$$

After some algebra, that equation yields $-h^2k + hk + hk^2 = h^2 + k^2$, hence

$$\begin{aligned}-h^2k + hk + hk^2 &\geq 2hk \\-h^2k + hk^2 &\geq hk \\-h + k &\geq 1 \\x + z = 2 - h + k &\geq 3 \\y = 3 - (x + z) &\leq 0\end{aligned}$$

—contradiction! Therefore $z \leq x$, so $x = y = z = 1$. That is, X , Y , and Z are the edge midpoints of $\triangle ABC$. ♦

Exercise 18. Using concepts and techniques introduced in sections 5.1 and 5.4, describe the center of gravity of the union of the edges of a triangle. You may assume that the center of gravity of a segment is its midpoint.

Solution. Replace edge $\overline{AB}$ of $\triangle ABC$ by a weight $c = AB$ at its midpoint C' . Replace the other edges similarly. Now replace the weights a at A' and b at B' by a weight $a + b$ at the point X on $\overline{A'B'}$ such that

$$\frac{XA'}{XB'} = \frac{b}{a}.$$

The bisector g of $\angle C'$ of the medial triangle $\triangle A'B'C'$ meets its edge $\overline{A'B'}$ at X , so the center of gravity of the system of weights lies on g . Similarly, it lies on the other angle bisectors of $\triangle A'B'C'$. Therefore, the center of gravity of the union of the edges of a triangle is the incenter of its medial triangle. ♦

Exercise 19. Finish the proof of Feuerbach's theorem begun in section 5.10 by showing that the Feuerbach circle of any triangle is tangent externally to its excircles.

Solution. Not available. ♦

Plane trigonometry

Several exercises under this heading complete the treatment of plane trigonometry started in section 3.13 and continued in 5.5. The last two exercises augment this chapter's list of arcane and entertaining formulas involving various triangle geometry concepts. They're just a sample of the huge stock to choose from. Exercise 20 connects the trigonometry in section 5.5 with Menelaus' theorem.

Exercise 20. Write Menelaus' product in terms of trigonometric functions of angles $\angle ACZ$, $\angle ZCB$, $\angle BAX$, $\angle XAC$, $\angle CBY$, and $\angle YBA$ in figure 5.4.1. How should you treat these angles consistently as positive or negative, so that the sign of the product is correct in all cases?

Solution. Choose a coordinate system. For noncollinear points P , V , and X , define the sign of $\angle PQR$ to be that of $\det[P - V, X - V]$. Note that $\angle ABC$, $\angle BCA$, and $\angle CAB$ all have the same sign. Choose scales on the edge lines of $\triangle ABC$ in order to use directed distances, but don't use directed distances on other lines. By the law of sines,

$$\frac{A \text{ to } Z}{Z \text{ to } B} = \frac{\frac{A \text{ to } Z}{CZ}}{\frac{Z \text{ to } B}{CZ}} = \frac{\frac{\sin m\angle ACZ}{\sin m\angle CAB}}{\frac{\sin m\angle ZCB}{\sin m\angle ABC}}$$

$$\frac{B \text{ to } X}{X \text{ to } C} = \frac{\frac{\sin m\angle BAX}{\sin m\angle ABC}}{\frac{\sin m\angle XAC}{\sin m\angle BCA}}, \quad \frac{C \text{ to } Y}{Y \text{ to } A} = \frac{\frac{\sin m\angle CBY}{\sin m\angle BCA}}{\frac{\sin m\angle YBA}{\sin m\angle CAB}}.$$

Thus, Menelaus' product is

$$\frac{\sin m\angle ACZ}{\sin m\angle ZCB} \frac{\sin m\angle BAX}{\sin m\angle XAC} \frac{\sin m\angle CBY}{\sin m\angle YBA} \cdot \blacklozenge$$

The next two problems summarize and provide examples for the part of a plane trigonometry course commonly called "solving triangles." They show how to determine the remaining side(s) or angle(s) of a triangle when you're given ASA, SSS, SAS, or SSA data. The last of these, often called the "ambiguous" case, is segregated from the others because SSA data sometimes don't determine the triangle uniquely. You made a preliminary study of that case in exercise 4.6.2.

Exercise 21, Part 1. Design an algorithm for computing the remaining angle measure and edge lengths of $\triangle ABC$ given ASA data $m\angle A$, $b =$

CA , and $m\angle C$. Clearly, any valid input must satisfy the conditions $0^\circ < m\angle A$, $m\angle C < 180^\circ$, and $0 < b$. Is any further condition necessary? If so, and you don't check it at the onset, how might your algorithm fail? Demonstrate your algorithm for the data sets

$m\angle A$	b	$m\angle C$
55°	65	75°
85°	95	105°

Part 2. Same problem for SSS data $a = BC$, b , and $c = BA$. Must valid input satisfy any conditions beyond the obvious $0 < a, b, c$? Try your algorithm on the data sets

a	b	c
70	65	82
70	152	82
70	153	82
148	65	82

Part 3. Same problem for SAS data a , $m\angle B$, and c . Must valid input satisfy any conditions beyond the obvious $0 < a, c$ and $0^\circ < m\angle B < 180^\circ$? Try your algorithm on the data set

a	$m\angle B$	c
70	50°	82

Solution, Part 1. If the input is valid, then

$$m\angle A + m\angle B + m\angle C = 180^\circ \quad m\angle B = 180^\circ - m\angle A - m\angle C$$

$$\frac{a}{\sin m\angle A} = \frac{b}{\sin m\angle B}$$

$$a = \frac{b \sin m\angle A}{\sin m\angle B} \quad c = \frac{b \sin m\angle C}{\sin m\angle B}.$$

Valid data must also satisfy the condition $m\angle A + m\angle C < 180^\circ$, else the $\angle B$ formula would yield a negative angle measure. For the first data set,

$$m\angle B = 180^\circ - 55^\circ - 75^\circ = 50^\circ$$

$$a = \frac{65 \sin 55^\circ}{\sin 50^\circ} \approx 70 \quad c = \frac{65 \sin 75^\circ}{\sin 50^\circ} \approx 82.$$

The second set is invalid because $m\angle A + m\angle C = 190^\circ > 180^\circ$.

Part 2. If the input is valid, then

$$b^2 = c^2 + a^2 - 2ca \cos m\angle B$$

$$\begin{aligned} m\angle B &= \arccos \frac{a^2 - b^2 + c^2}{2ca} & m\angle C &= \arccos \frac{a^2 + b^2 - c^2}{2ab} \\ m\angle A &= 180^\circ - m\angle B - m\angle C. \end{aligned}$$

Valid data must also satisfy the conditions $b < c + a$, $c < a + b$, and $a < b + c$. The invalid condition $b \geq c + a$ is equivalent to $b^2 \geq (c + a)^2$: that is, to $a^2 - b^2 + c^2 \leq -2ca$, or

$$\frac{a^2 - b^2 + c^2}{2ca} \leq -1.$$

Therefore, this condition corresponds to $m\angle B = 180^\circ$ or to failure of the inverse cosine function. The invalid condition $c \geq a + b$ or $a \geq b + c$ is equivalent to $|a - c| \geq b$, that is, $(a - c)^2 \geq b^2$, or

$$\frac{a^2 - b^2 + c^2}{2ca} \geq +1.$$

Therefore, this condition corresponds to $m\angle B = 0^\circ$ or to another failure of the inverse cosine function. For the first data set,

$$\begin{aligned} m\angle B &= \arccos \frac{70^2 - 65^2 + 82^2}{2 \cdot 82 \cdot 70} \approx 49.9^\circ \\ m\angle C &= \arccos \frac{70^2 + 65^2 - 82^2}{2 \cdot 70 \cdot 65} \approx 74.7^\circ \\ m\angle A &= 180^\circ - m\angle B - m\angle C \approx 55.4^\circ. \end{aligned}$$

The second and third data sets are invalid because $b \geq a + c$; the fourth, because $a > b + c$.

Part 3. If the input is valid, then

$$b = \sqrt{c^2 + a^2 - 2ca \cos m\angle B}.$$

The radicand is always positive, so no additional condition is necessary. Once you know b , you can use the SSS algorithm

$$\begin{aligned} m\angle A &= \arccos \frac{-a^2 + b^2 + c^2}{2bc} = \arccos \frac{c - a \cos m\angle B}{b} \\ m\angle C &= 180^\circ - m\angle A - m\angle B. \end{aligned}$$

For the given data set,

$$\begin{aligned} b &= \sqrt{82^2 + 70^2 - 2 \cdot 70 \cdot 82 \cos 50^\circ} \approx 65.2. \\ m\angle A &= \arccos \frac{82 - 70 \cos 50^\circ}{b} \approx 55.4^\circ \\ m\angle C &= 180^\circ - m\angle A - m\angle B \approx 74.6^\circ. \blacklozenge \end{aligned}$$

Exercise 22. Design an algorithm for computing all possible values of the remaining angle measure and edge lengths of $\triangle ABC$ given SSA data $a = BC$, $b = CA$, and $m\angle A$. According to exercise 4.6.2 there are at most two possible solution sets $m\angle B$, $m\angle C$, and $c = AB$. Clearly, any valid input must satisfy the conditions $0 < a, b$ and $0^\circ < m\angle A < 180^\circ$. Is any further condition necessary? If so, and you don't check it at the onset, how might your algorithm fail? Demonstrate your algorithm for the data sets

a	b	$m\angle A$
80	90	100°
90	80	100°
80	80	80°
30	60	30°
30	60	28°
28	60	28°

Solution. If $\angle A$ is right or obtuse, it must be the largest angle, so $a = \overline{BC}$ must be the longest edge, and the data must satisfy $a > b$. In that case, there's just one solution:

$$\frac{\sin m\angle B}{b} = \frac{\sin m\angle A}{a} \quad m\angle B = \arcsin \frac{b \sin m\angle A}{a}$$

$$m\angle C = 180^\circ - m\angle A - m\angle B$$

$$\frac{c}{\sin m\angle C} = \frac{a}{\sin m\angle A} \quad c = \frac{a \sin m\angle C}{\sin m\angle A}.$$

If you tried to carry out this computation under the invalid condition $m\angle A \geq 90^\circ$ and $a \leq b$ —for example, with data set (1)—you'd get $(b/a) \sin m\angle A \geq \sin m\angle A$, so either the inverse sine would fail, or you'd compute $m\angle B > \arcsin \sin m\angle A = 180^\circ - m\angle A$ and that would yield the obvious error $m\angle C < 0^\circ$. For data set (2), this computation yields

$$m\angle B = \arcsin \frac{80 \sin 100^\circ}{90} \approx 61^\circ$$

$$m\angle C = 180^\circ - 100^\circ - m\angle B \approx 19^\circ$$

$$c = \frac{90 \sin m\angle C}{\sin 100^\circ} \approx 30.$$

From here on, suppose $\angle A$ is acute. If $a \geq b$, there's just one solution and $m\angle A \geq m\angle B$. Thus $\angle B$ must be acute, and the previous paragraph's procedure will yield the solution. For data set (3) it's

$$m\angle B = \arcsin \frac{80 \sin 80^\circ}{80} = 80^\circ$$

$$m\angle C = 180^\circ - 80^\circ - 80^\circ = 20^\circ$$

$$c = \frac{80 \sin 20^\circ}{\sin 80^\circ} \approx 28.$$

Now let $d = b \sin m\angle A$. If $a = d$, there's just one solution: $m\angle B = 90^\circ$, $m\angle C = 90^\circ - m\angle A$, and $c = \sqrt{b^2 - a^2}$. Data set (4) satisfies this condition, and

$$m\angle B = 90^\circ$$

$$m\angle C = 90^\circ - 30^\circ = 60^\circ$$

$$c = \sqrt{60^2 - 30^2} \approx 52.$$

If $a > d$, there are exactly two solutions. Note first that

$$\frac{\sin m\angle B}{b} = \frac{\sin m\angle A}{a}.$$

Now let $m_1 = \arcsin \frac{b \sin m\angle A}{a}$, so that $m\angle B = m_1 < 90^\circ$ or $m\angle B = 180^\circ - m_1 > 90^\circ$ and

$$m\angle C = 180^\circ - m\angle A - m\angle B$$

$$c = \frac{a \sin m\angle C}{\sin m\angle A}.$$

Data set (5) satisfies this condition, and

$$m_1 = \arcsin \frac{60 \sin 28^\circ}{30}$$

$$m\angle B = m_1 \approx 70^\circ \text{ or } 110^\circ \approx 180^\circ - m_1$$

$$m\angle C = 180^\circ - 28^\circ - m\angle B \approx 82^\circ \text{ or } 42^\circ$$

$$c = \frac{30 \sin m\angle C}{\sin 28^\circ} \approx 63 \text{ or } 43.$$

If you tried to carry out this computation under the invalid condition $m\angle A < 90^\circ$ and $a < b \sin m\angle A$ —for example, with data set (6) —you'd get $(b/a) \sin m\angle A > 1$, and the inverse sine function would fail. ♦

Exercise 23. Derive algebraically the formulas for $\tan(\alpha \pm \beta)$ in terms of $\tan \alpha$ and $\tan \beta$, and the formula for $\tan 2\alpha$ in terms of $\tan \alpha$. Derive a formula for $\tan^{1/2} \alpha$ in terms of $\sin \alpha$ and $\cos \alpha$ that doesn't require a $\pm$ sign.

Solution.

$$\tan(\alpha \pm \beta) = \frac{\sin(\alpha \pm \beta)}{\cos(\alpha \pm \beta)} = \frac{\sin \alpha \cos \beta \pm \cos \alpha \sin \beta}{\cos \alpha \cos \beta \mp \sin \alpha \sin \beta}$$

$$\begin{aligned}
&= \frac{\frac{\sin \alpha \cos \beta}{\cos \alpha \cos \beta} \pm \frac{\cos \alpha \sin \beta}{\cos \alpha \cos \beta}}{\frac{\cos \alpha \cos \beta}{\cos \alpha \cos \beta} \mp \frac{\sin \alpha \sin \beta}{\cos \alpha \cos \beta}} = \frac{\frac{\sin \alpha}{\cos \alpha} \pm \frac{\sin \beta}{\cos \beta}}{1 \mp \frac{\sin \alpha}{\cos \alpha} \frac{\sin \beta}{\cos \beta}} \\
&= \frac{\tan \alpha \pm \tan \beta}{1 \mp \tan \alpha \tan \beta} \\
\tan 2\alpha &= \frac{\tan \alpha + \tan \alpha}{1 - \tan \alpha \tan \alpha} = \frac{2 \tan \alpha}{1 - \tan^2 \alpha} \\
\sin \alpha &= 2 \sin^{1/2} \alpha \cos^{1/2} \alpha = \frac{\sin^{1/2} \alpha}{\cos^{1/2} \alpha} \cdot 2 \cdot \cos^{1/2} \alpha \\
&= \frac{\sin^{1/2} \alpha}{\cos^{1/2} \alpha} \cdot 2 \cdot \frac{1 + \cos \alpha}{2} \\
\frac{\sin \alpha}{1 + \cos \alpha} &= \tan^{1/2} \alpha. \blacklozenge
\end{aligned}$$

Exercise 24. Show that for $n = 0, 1, 2, \dots$ there's a polynomial T_n of degree n —called a *Chebyshev* polynomial—such that for all x , $\cos nx = T_n(\cos x)$. Derive the *recursion* formula $T_{n+1}(y) = 2yT_n(y) - T_{n-1}(y)$. Write out the formulas for T_0 to T_5 , sketch their graphs, and describe their roots, local extrema, and symmetry. What happens if you try something like this for $\sin nx$?

Solution. Define $T_0(y) = 1$ and $T_1(y) = y$ for all y , so that $T_0(\cos x) = \cos(0x)$ and $T_1(\cos x) = \cos(1x)$ for all x . Thus, the statement

$\mathcal{P}_m$: there's a polynomial T_m of degree m such that $T_m(\cos x) = \cos mx$ for all x

is true for $m = 0, 1$. Now let n be an integer > 1 and suppose $\mathcal{P}_m$ is true for all $m \leq n$. The following argument shows that $\mathcal{P}_{n+1}$ is true also. First, note that

$$\begin{aligned}
&\cos(n+1)x + \cos(n-1)x \\
&= (\cos nx \cos x - \sin nx \sin x) + (\cos nx \cos x + \sin nx \sin x) \\
&= 2 \cos nx \cos x \\
&\cos(n+1)x = 2 \cos nx \cos x - \cos(n-1)x \\
&= 2 T_n(\cos x) \cos x - T_{n-1}(\cos x).
\end{aligned}$$

Now define $T_{n+1}(y) = 2yT_n(y) - T_{n-1}(y)$, a polynomial of degree $n+1$. Clearly,

$$\cos(n+1)x = T_{n+1}(\cos x).$$

Therefore, $\mathcal{P}_m$ is true for $m = n + 1$. By recursion, it's true for all $m = 0, 1, 2, \dots$. Finally,

$$\begin{aligned} T_2(y) &= 2yT_1(y) - T_0(y) \\ &= 2y^2 - 1 \\ T_3(y) &= 2yT_2(y) - T_1(y) = 2y(2y^2 - 1) - y \\ &= 4y^3 - 3y \\ T_4(y) &= 2yT_3(y) - T_2(y) = 2y(4y^3 - 3y) - (2y^2 - 1) \\ &= 8y^4 - 8y^2 + 1 \\ T_5(y) &= 2yT_4(y) - T_3(y) = 2y(8y^4 - 8y^2 + 1) - (4y^3 - 3y) \\ &= 16y^5 - 20y^3 + 5y. \end{aligned}$$

A more detailed look at this recursion shows that T_n is even when n is even and odd when n is odd.

For $k = 0, \dots, n - 1$ let $x_k = (2^{k+1/2n})\pi$ and $y_k = \cos x_k$. Then $0 < x_0 < \dots < x_{n-1} < \pi$, so $y_0 > \dots > y_{n-1}$. Moreover, $T_n(y_k) = \cos nx_k = \cos(k + 1/2)\pi = 0$. Since T_n has degree n , it has at most n roots: just $y_0, \dots, y_{n-1}$. The roots of its derivative T_n' lie between these values and yield alternate local minimum and maximum values for T_n .

For $k = 1, \dots, n - 1$ let $w_k = k/n \pi$ and $z_k = \cos w_k$. Then $0 < w_1 < \dots < w_{n-1} < \pi$, so $z_1 > \dots > z_{n-1}$. Moreover,

$$\begin{aligned} \left. \frac{d}{dx} T_n(\cos x) \right|_{x=w_k} &= T_n'(\cos x)(-\sin x)|_{x=w_k} = -T_n'(z_k) \sin w_k \\ &= \left. \frac{d}{dx} \cos nx \right|_{x=w_k} = -n \sin nw_k = -n \sin k\pi = 0. \end{aligned}$$

Since $0 < w_k < \pi$, $\sin w_k \neq 0$, so $T_n'(z_k) = 0$. The local extrema of T_n occur at $z_1, \dots, z_{n-1}$.

You could make a similar argument with $\sin x$ instead of $\cos x$ for odd n , but not for even n , because you can't express $\sin(2nx)$ as a polynomial in $\sin x$. ♦

Exercise 25. Let r_a , r_b , and r_c be the exradii of ΔABC ; r and R , its inradius and circumradius; Δ its area; s its semiperimeter; and h_a , h_b , and h_c be the lengths of its altitudes. Prove that

$$\begin{aligned} \frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c} &= \frac{1}{r} = \frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c} & r r_a r_b r_c &= \Delta^2 \\ r_a + r_b + r_c &= 4R + r & r_a r_b + r_b r_c + r_c r_a &= s^2. \end{aligned}$$

Solution. By theorems 5.9.4, 3.8.9, Hero's formula, and theorem 5.5.11,

$$\frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c} = \frac{s-a}{\Delta} + \frac{s-b}{\Delta} + \frac{s-c}{\Delta} = \frac{3s-2s}{\Delta} = \frac{s}{\Delta} = \frac{1}{r}$$

$$\begin{aligned}
\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c} &= \frac{a}{2\Delta} + \frac{b}{2\Delta} + \frac{c}{2\Delta} = \frac{s}{\Delta} = \frac{1}{r} \\
r r_a r_b r_c &= \frac{\Delta}{s} \frac{\Delta}{s-a} \frac{\Delta}{s-b} \frac{\Delta}{s-c} = \frac{\Delta^4}{\Delta^2} = \Delta^2 \\
r_a + r_b + r_c - r &= \frac{\Delta}{s-a} + \frac{\Delta}{s-b} + \frac{\Delta}{s-c} - \frac{\Delta}{s} \\
&= \Delta \frac{s(s-b)(s-c) + s(s-c)(s-a) + s(s-a)(s-b) - (s-a)(s-b)(s-c)}{s(s-a)(s-b)(s-c)} \\
&= \frac{s(s-c)[(s-b) + (s-a)] + (s-a)(s-b)[s - (s-c)]}{\Delta} \\
&= \frac{s(s-c)c + (s-a)(s-b)c}{\Delta} \\
&= \frac{c}{\Delta} [s^2 - cs + s^2 - as - bs + ab] = \frac{abc}{\Delta} = 4R.
\end{aligned}$$

By the first equation displayed in this solution,

$$\begin{aligned}
\frac{r_b r_c + r_c r_a + r_a r_b}{r_a r_b r_c} &= \frac{1}{r} \\
r_a r_b + r_b r_c + r_c r_a &= \frac{r_a r_b r_c}{r} = \frac{r r_a r_b r_c}{r^2} = \frac{\Delta^2}{r^2} = s^2. \diamond
\end{aligned}$$

Exercise 26. Prove that the sum of the cosines of the measures of the angles of a triangle equals 1 plus the ratio of its inradius to its circumradius.

*Solution.*⁵ Use the standard notation with $\triangle ABC$. Let Δ be its area and $\sigma = \cos m\angle A + \cos m\angle B + \cos m\angle C$. By the law of cosines,

$$\begin{aligned}
\sigma &= \frac{-a^2 + b^2 + c^2}{2bc} + \frac{a^2 - b^2 + c^2}{2ca} + \frac{a^2 + b^2 - c^2}{2ab} \\
&= \frac{a(-a^2 + b^2 + c^2) + b(a^2 - b^2 + c^2) + c(a^2 + b^2 - c^2)}{2abc}.
\end{aligned}$$

Expand the numerator, arrange its terms in descending order of powers of a , then b , then c , and divide it by $a + b - c = 2(s - c)$. You get quotient

$$-a^2 + 2ab - b^2 + c^2 = c^2 - (a - b)^2$$

⁵ This solution is due to the author's students Peter Kohn (Spring 1983) and Jayme Goodale (Spring 1992).

$$= (-a + b + c)(a - b + c) = 4(s - a)(s - b)$$

and remainder $2abc$. By Hero's formula and theorems 5.5.11 and 5.9.4,

$$\begin{aligned}\sigma &= \frac{4(s-a)(s-b) \cdot 2(s-c) + 2abc}{2abc} \\ &= 1 + \frac{4s(s-a)(s-b)(s-c)}{sabc} = 1 + \frac{4\Delta^2}{sabc} = 1 + 4\Delta \frac{\Delta}{sabc} \\ &= 1 + \frac{abc}{R} \frac{\Delta}{sabc} = 1 + \frac{\Delta}{Rs} = 1 + \frac{rs}{Rs} = 1 + \frac{r}{R}. \blacklozenge\end{aligned}$$

Exercise 27 shows how to approximate $\sin 1^\circ$ accurately, using theorem 5.5.12 and values of $\sin 1^{1/2}^\circ$ and $\sin 3/4^\circ$ computed via subtraction and half angle formulas. Van der Waerden reports⁶ that Aristarchus of Samos first used this technique around 280 B.C. to approximate $\sin 3^\circ$. About A.D. 150 the Alexandrian astronomer Claudius Ptolemy employed it systematically to compute what amounts to a table of values $\sin \theta$ for $\theta = 1^\circ, 2^\circ, 3^\circ, \dots$. The method should remind you of real analysis arguments. That isn't surprising, since theorem 5.5.12 is an important step in justifying the formula for the derivative of the sine function.

Exercise 27, Part 1. Derive algebraic formulas for $\sin 1^{1/2}^\circ$ and $\sin 3/4^\circ$ and check them with your calculator.

Part 2. Show that $\sin \alpha - \sin \beta = 2 \cos \mu \sin \nu$, where $\mu = 1/2(\alpha + \beta)$ and $\nu = 1/2(\alpha - \beta)$.

Part 3. Suppose $0^\circ < \beta < \alpha < 90^\circ$. Prove that

$$2 \cos \mu \sin \nu < \frac{\alpha - \beta}{\beta} \sin \beta.$$

Part 4. Combine parts (2) and (3) to show that when $0^\circ < \gamma < \beta < \alpha < 90^\circ$,

$$\frac{\beta}{\alpha} \sin \alpha < \sin \beta < \frac{\beta}{\gamma} \sin \gamma.$$

Part 5. Use $\alpha, \beta, \gamma = 1^{1/2}^\circ, 1^\circ, 3/4^\circ$ in part (4), with the values you calculated in part (1), to estimate $\sin 1^\circ$. What's the maximum possible error?

Solution, Part 1. Use $\sin 30^\circ = 1/2$, $\cos 36^\circ = 1/4(1 + \sqrt{5})$ (see exercise 4.9.1), the Pythagorean identity, and the cosine difference and half angle formulas to derive

$$\cos 1^{1/2}^\circ = \frac{1}{2} \sqrt{2 + \frac{1}{2} \sqrt{8 + \sqrt{3} + \sqrt{15} + \sqrt{10 - 2\sqrt{5}}}} \approx 0.999657325$$

⁶ 1963, Chapter VII.

$$\cos 3/4^\circ = \sqrt{\frac{1 + \cos 1/2^\circ}{2}} \approx 0.999914327.$$

Part 2. Use the sine addition and subtraction formulas and algebra.

Part 3. First, $\mu > \beta$. Second, by theorem 5.5.12, $\sin \nu < (\pi/180^\circ) \nu$ and $(\pi/180^\circ) \beta < \tan \beta$. Therefore,

$$\begin{aligned} 2 \cos \mu \sin \nu &< 2 \cos \beta \sin \nu < 2(\cos \mu) \frac{\pi \nu}{180^\circ} = \frac{2\nu}{\beta} \frac{\pi \beta}{180^\circ} \cos \beta \\ &< \frac{2\nu}{\beta} \tan \beta \cos \beta = \frac{2\nu}{\beta} \cos \beta = \frac{\alpha - \beta}{\beta} \sin \beta. \end{aligned}$$

Part 4. From $\beta < \alpha$ and part (3) you get

$$\sin \alpha - \sin \beta < \frac{\alpha - \beta}{\beta} \sin \beta,$$

which entails $\frac{\beta}{\alpha} \sin \alpha < \sin \beta$. The other inequality is proved analogously.

Part 5. With the suggested arguments α , β , and γ , part (4) yields

$$0.017451298 \approx 2/3 \sin 1/2^\circ < \sin 1^\circ < 4/3 \sin 3/4^\circ \approx 0.017452794.$$

So $\sin 1^\circ \approx 0.017452$ with error $< 10^{-6}$. ♦

Spherical trigonometry

The exercises under this heading complete the survey of spherical trigonometry begun in the section 4.9 exercises. For a century, this material has been a mathematical backwater. In most geometry texts, you could only find it briefly mentioned as a contrast to the familiar plane Euclidean geometry. Its practical aspects were confined to applications texts: astronomy, geodesy, navigation. Its applications now reach into broader areas as we learn to write software to recognize objects in a three-dimensional environment and to control machines that move them about and/or display their images. These exercises generally follow the treatment in Marcel Berger's terse but penetrating monograph.⁷

Exercise 28, Part 1. Prove the law of sines for a spherical triangle $\triangle ABC$:

$$\frac{\sin m\angle A}{\sin m\widehat{BC}} = \frac{\sin m\angle B}{\sin m\widehat{CA}} = \frac{\sin m\angle C}{\sin m\widehat{AB}}.$$

⁷ Berger 1987, section 18.6.

Part 2. Prove Menelaus' theorem for a spherical $\triangle ABC$: If X, Y, Z are points on the great circles containing $\widehat{BC}, \widehat{CA}, \widehat{AB}$, and X, Y, Z lie on a great circle, then

$$\frac{\sin m\widehat{AZ}}{\sin m\widehat{ZB}} \frac{\sin m\widehat{BX}}{\sin m\widehat{XC}} \frac{\sin m\widehat{CY}}{\sin m\widehat{YA}} = 1.$$

Solution, Part 1. Let O be the center of the sphere and ε be the plane perpendicular to OBC through O and A . Let B' be the point opposite B on the sphere, and D be the intersection of ε with the great semicircle $\widehat{BB'}$ that contains C . Consider spherical triangles $\triangle ADB$ and $\triangle ADC$, which both have $m\angle D = 90^\circ$. By exercise 4.9.4, $\sin m\angle B \sin m\widehat{AB} = \sin m\widehat{AD} = \sin m\angle C \sin m\widehat{AC}$. This yields half the law of sines; the other half follows similarly.

Part 2. Use the law of sines. The “Menelaus product” is

$$\begin{aligned} & \frac{\sin m\widehat{AZ}}{\sin m\widehat{YA}} \frac{\sin m\widehat{BX}}{\sin m\widehat{ZB}} \frac{\sin m\widehat{CY}}{\sin m\widehat{XC}} \\ &= \frac{\sin m\angle AYZ}{\sin m\angle YZA} \frac{\sin m\angle BZX}{\sin m\angle ZXB} \frac{\sin m\angle CXY}{\sin m\angle XYC} \\ &= \frac{\sin m\angle AYZ}{\sin m\angle XYC} \frac{\sin m\angle BZX}{\sin m\angle YZA} \frac{\sin m\angle CXY}{\sin m\angle ZXB} = 1 \cdot 1 \cdot 1 = 1. \blacklozenge \end{aligned}$$

Exercise 28 suggests some questions that would make a good project of larger scope. This form of Menelaus' theorem doesn't use “directed” arcs as its plane form uses directed distances. All terms in the Menelaus product are sines, so it comes out $+1$ instead of -1 as in the plane. Is it possible to make such distinctions consistently on the sphere? Is there a converse of this form of Menelaus' theorem? Is there a spherical form of Ceva's theorem?

Exercise 29 is mainly a problem in visualization. Parts 1 and 2 require careful reasoning about the sides of various planes. In part 2 you should concentrate on the plane ε through O perpendicular to $\vec{OB}$, and its intersections with planes OAB and OBC . The only computation you need for those parts is angle addition.

Exercise 29, Part 1. Consider a spherical triangle $\triangle ABC$ on a sphere Σ with center O . The perpendicular to OBC through O intersects Σ at two points. One lies on the same side of OBC as A ; call it A' . Define B' and C' similarly. Show that A', B' , and C' aren't coplanar with O . Thus they form a spherical $\triangle A'B'C'$ called the *polar* triangle of $\triangle ABC$. Show that $\triangle ABC$ is in turn the polar triangle of $\triangle A'B'C'$.

Part 2. Let a', b' , and c' be the measures of the arcs of the polar triangle opposite A', B' , and C' . Show that

$$a' = 180^\circ - m\angle A \quad m\angle A' = 180^\circ - a$$

and derive similar formulas for the measures of its other arcs and angles.

Part 3. Let a , b , and c be the measures of the arcs of $\triangle ABC$ opposite A , B , and C . Show that

$$\begin{aligned} a &< b + c & m\angle A + m\angle B &< m\angle C + 180^\circ \\ b &< c + a & m\angle B + m\angle C &< m\angle A + 180^\circ \\ c &< a + b & m\angle C + m\angle A &< m\angle B + 180^\circ. \\ a + b + c &< 360^\circ \end{aligned}$$

Part 4. Consider the following problem in more detail: For a vertex V of the spherical triangle, which intersection of Σ with $\vec{OV'}$ should be called V' ? Do parts (1) to (3) fail if you make one or more of these choices differently? What if you set up a coordinate system with origin O , made the sphere's radius 1, and chose $A' = B \times C$, etc.?

Solution, Part 1. From the definitions of A' , B' and C' ,

$$\begin{aligned} \vec{OA'} \perp \vec{OB}, \vec{OC} & \quad A \text{ and } A' \text{ lie on the same side of } OBC, \\ \vec{OB'} \perp \vec{OC}, \vec{OA} & \quad B \text{ and } B' \text{ lie on the same side of } OCA, \\ \vec{OC'} \perp \vec{OA}, \vec{OB} & \quad C \text{ and } C' \text{ lie on the same side of } OAB. \end{aligned}$$

If A' , B' , and C' were coplanar with O , then $\vec{OA'} \perp \vec{OA}, \vec{OB}, \vec{OC}$, so A , B , and C would be coplanar with O —contradiction! Note that $OCA, OC'A' \perp OBB'$. Define

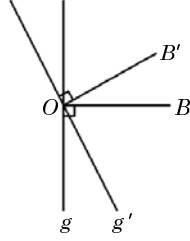
$$\begin{aligned} g &= OBB' \cap OCA \\ g' &= OBB' \cap OC'A'. \end{aligned}$$

Then $g \perp \vec{OB'}$, $g' \perp \vec{OB}$, and B and B' lie on the same side of g in OBB' . As shown in Figure 5.11S.5, B and B' lie on the same side of g' , hence on the same side of $OC'A'$. It follows that $B = (B')'$. Similarly, $C = (C')'$ and $A = (A')'$.

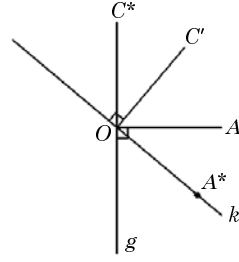
Part 2. The plane $\varepsilon \perp \vec{OB}$ through O passes through A' and C' . Figure 5.11S.6 depicts the case where $\angle A'OC'$ is acute; in figure 5.11S.7, it's obtuse. Let $g = \varepsilon \cap OBC$ and $k = \varepsilon \cap OAB$, so that $g \perp \vec{OA'}$ and $k \perp \vec{OC'}$. Let A^* be any point of k on the same side of $\vec{OB}$ in OAB as A , and C^* be any point of g on the same side of $\vec{OB}$ in OBC as C . Then A^* and C^* lie on the same sides of g and k in ε as A' and C' . Now, $\angle B$ belongs to the dihedral angle formed by the A side of $\vec{OB}$ in OAB and the C side of $\vec{OB}$ in OBC . Therefore, $b' = m\angle A'OC' =$

$$\begin{aligned} 90^\circ - m\angle C'OC^* &= 90^\circ - (m\angle A^*OC^* - 90^\circ) \quad \text{in figure 5.11S.6, or} \\ 90^\circ + m\angle C'OC^* &= 90^\circ + (90^\circ - m\angle A^*OC^*) \quad \text{in figure 5.11S.7.} \end{aligned}$$

Each of these equals $180^\circ - m\angle B$. If $\angle A'OC'$ is right, then you can take $A^* = A'$ and $C^* = C'$ and get the same result. Now apply it to the polar

**Figure 5.11S.5**

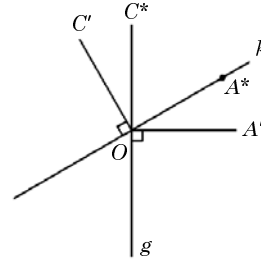
Exercise 29 solution,
Part (1): plane $OB B'$

**Figure 5.11S.6**

Exercise 29 solution,
Part (2): acute $\angle A'OC'$

Figure 5.11S.7

Exercise 29 solution,
part (2): obtuse $\angle A'OC'$



triangle to get $b = b'' = 180^\circ - m\angle B'$. You can derive the remaining results similarly.

Part 3. Inequalities $a < b + c$, $b < c + a$, and $c < a + b$ are straightforward elementary geometry exercises. Apply them to the polar triangle:

$$\begin{aligned} m\angle A + m\angle B &= (180^\circ - a') + (180^\circ - b') \\ &= 360^\circ - (a' + b') < 360^\circ - c' = 360^\circ - (180^\circ - m\angle C). \end{aligned}$$

$$\begin{aligned} a + b + c &= (180^\circ - m\angle A') + (180^\circ - m\angle B') + (180^\circ - m\angle C') \\ &= 540^\circ - (m\angle A' + m\angle B' + m\angle C') < 540^\circ - 180^\circ. \end{aligned}$$

Part 4. Not available. ♦

Exercise 30. Prove the *cosine laws* for a spherical triangle ΔABC : If a , b , and c are the measures of the arcs of ΔABC opposite A , B , and C , then

$$\begin{aligned} \cos a &= \cos b \cos c + \sin b \sin c \cos m\angle A, \\ \cos m\angle A &= -\cos m\angle B \cos m\angle C + \sin m\angle B \sin m\angle C \cos a. \end{aligned}$$

Suggestion: Prove the first equation geometrically. You may have to consider the case $c = 90^\circ$ separately. For the second, apply exercise 29.

Solution. Find the foot P of the perpendicular to $\vec{OA}$, and the line $h \perp \vec{OA}$ in OAB through P . Assume h is neither equal nor parallel to $\vec{OB}$; the opposite case is considered later. Then $h \cap \vec{OB}$ is a point R , $m\angle CPR = m\angle A$, and $m\angle COR = \alpha$. Apply the plane law of cosines to $\triangle CPR$ and $\triangle COR$:

$$\begin{aligned}(RC)^2 &= (PC)^2 + (PR)^2 - 2(PC)(PR) \cos m\angle A \\ (RC)^2 &= (OC)^2 + (OR)^2 - 2(OC)(OR) \cos \alpha.\end{aligned}$$

Subtract these equations. Some algebra yields

$$\begin{aligned}\cos \alpha &= \frac{OP}{OC} \frac{OP}{OR} + \frac{PC}{OC} \frac{PR}{OR} \cos m\angle A \\ &= \cos b \cos c + \sin b \sin c \cos m\angle A,\end{aligned}$$

the first form of the spherical law of cosines.

Now suppose h is equal or parallel to $\vec{OB}$, so that $c = 90^\circ$. If $h = \vec{OB}$, then also $b = 90^\circ$, $m\angle A = \alpha$, and the first form of the law holds. Assume $h \neq \vec{OB}$. Find the foot Q of the perpendicular from C to h . If $Q = P$, then $\alpha = b = 90^\circ$ and the desired equation holds. Assume $Q \neq P$. Find the point R on $\vec{OB}$ so that $OPQR$ is a rectangle. Then $CQR \perp h$, so $CQR \perp \vec{OB}$. Moreover,

$$\begin{aligned}\sin b \cos m\angle A &= \sin m\angle COA \cos m\angle CPQ \\ &= (PC/OC)(OR/PC) = OR/OC \\ &= \cos m\angle COB = \cos \alpha,\end{aligned}$$

the desired equation.

To prove the second form of the law, apply the first form to the polar triangle:

$$\begin{aligned}-\cos m\angle A &= \cos(180^\circ - \alpha') = \cos \alpha' \\ &= \cos b' \cos c' + \sin b' \sin c' \cos m\angle A' \\ &= \cos(180^\circ - m\angle B) \cos(180^\circ - m\angle C) \\ &\quad + \sin(180^\circ - m\angle B) \sin(180^\circ - m\angle C) \cos(180^\circ - \alpha) \\ &= (-\cos m\angle B)(-\cos m\angle C) + \sin m\angle B \sin m\angle C (-\cos \alpha) \\ &= \cos m\angle B \cos m\angle C - \sin m\angle B \sin m\angle C \cos \alpha. \blacklozenge\end{aligned}$$

Exercise 31. Imitate exercises 21 and 22 for spherical triangles. Include discussions of SAA and AAA data, too.

Solution. Not available. $\blacklozenge$

Exercise 32, Part 1. From the following data, find the airline distances from San Francisco to Bucharest and Christchurch. Assume that the earth is a sphere with radius $r = 3959$ miles.

<i>City</i>	<i>Latitude</i>	<i>Longitude</i>
San Francisco	37°46' N	122°25' W
Bucharest	44°25' N	26°07' E
Christchurch	43°33' S	172°40' E

How far have *you* been from home?

Part 2. Find the fraction of the earth's area covered by the spherical triangle formed by these cities. Describe the triangle by listing a few places on or near each of its edges.

Solution, Part 1. With the North Pole N , San Francisco and Bucharest form a spherical triangle $\triangle NSB$ with opposite arcs measuring n , s , and b :

$$\begin{aligned} m\angle N &= 122^\circ 25' + 26^\circ 07' = 148^\circ 32' \\ s &= 90^\circ - 44^\circ 25' = 45^\circ 35' \\ b &= 90^\circ - 37^\circ 46' = 52^\circ 14'. \end{aligned}$$

By the spherical law of cosines,

$$\begin{aligned} \cos n &= \cos s \cos b + \sin s \sin b \cos m\angle N \approx -0.0530 \\ n &\approx 93.04^\circ \\ \text{airline distance from } S \text{ to } B &= n\pi r/180^\circ \approx 6428 \text{ miles.} \end{aligned}$$

With N , San Francisco and Christchurch form a spherical triangle $\triangle NSC$ with opposite arcs measuring n , s , and c :

$$\begin{aligned} m\angle N &= 360^\circ - (122^\circ 25' + 172^\circ 40') = 64^\circ 55' \\ s &= 90^\circ + 43^\circ 33' = 133^\circ 33' \\ c &= 90^\circ - 37^\circ 46' = 52^\circ 14'. \end{aligned}$$

$$\begin{aligned} \cos n &= \cos s \cos c + \sin s \sin c \cos m\angle N \approx -0.1791 \\ n &\approx 100.32^\circ \\ \text{airline distance from } S \text{ to } C &= n\pi r/180^\circ \approx 6932 \text{ miles.} \end{aligned}$$

Part 2. With N , Bucharest and Christchurch form a spherical triangle $\triangle NBC$ with opposite arcs measuring n , b , and c :

$$\begin{aligned} m\angle N &= 172^\circ 40' - 26^\circ 07' = 146^\circ 33' \\ b &= 90^\circ + 43^\circ 33' = 133^\circ 33' \\ c &= 90^\circ - 44^\circ 25' = 45^\circ 35' \end{aligned}$$

$$\begin{aligned} \cos n &= \cos b \cos c + \sin b \sin c \cos m\angle N \approx -0.9141 \\ n &\approx 156.08^\circ \\ \text{airline distance from } B \text{ to } C &= n\pi r/180^\circ \approx 10785 \text{ miles.} \end{aligned}$$

Now consider the spherical triangle $\triangle BCS$: from part (1),

$$b \approx 100.32^\circ \quad c \approx 93.04^\circ \quad s \approx 156.08^\circ.$$

By the spherical law of cosines,

$$m\angle B = \arccos \frac{\cos b - \cos s \cos c}{\sin s \sin c} \approx 124.2^\circ.$$

Similarly,

$$m\angle C \approx 122.9^\circ \quad m\angle S \approx 160.1^\circ.$$

The area is then

$$\left(\frac{m\angle B + m\angle C + m\angle S}{180^\circ} - 1 \right) \pi r^2$$

so the ratio is

$$\frac{100\%}{4\pi} \left(\frac{m\angle B + m\angle C + m\angle S}{180^\circ} - 1 \right) \approx 31.6\%.$$

The great circle route from Bucharest to Christchurch crosses the Black Sea, passes near Yerevan, proceeds across the Caspian Sea and the Iranian and Afghan deserts to Qandahar, crosses central Pakistan, passes near Nagpur in central India, crosses the Bay of Bengal and the length of Sumatra, passes near Jakarta, crosses the Timor Sea and the Australian desert, passes near Canberra, and finally crosses the Tasman Sea to New Zealand's South Island.

From Christchurch to San Francisco, you fly over water, passing over the Cook Islands and crossing the equator near Christmas Island.

Returning from San Francisco to Bucharest, you pass over eastern Oregon, near Calgary, over Canada's Northwest Territories and Baffin Island, over central Greenland and the Norwegian Sea, and pass near Oslo, Linköping, and Warsaw to Romania. ♦

Analytic geometry

In the section 4.10 exercises, you constructed a tool kit of analytic formulas and methods corresponding to many synthetic techniques developed in chapter 3. You were asked to avoid vector cross products, but to use solutions of linear systems instead.

Exercise 33. Restudy those tools now in the light of section 5.6, and employ cross products and determinants where appropriate to streamline them. (This is the main use of cross products: given vectors X and Y , to find a vector Z such that $X \cdot Z = 0 = Y \cdot Z$.)

Solution. The suggested solutions to tasks 4, 10, and 14 in exercises 4.10.1 and 4.10.2 and to exercise 4.10.6, part 1, can be streamlined as follows.

Task 4. Points P, Q, R are collinear just in case $\det[PQR] = 0$.

Task 10. Distinct lines with equations $X = At + B$ and $X = Ct + D$ are parallel just in case $A \times C = 0$.

Task 14. Given intersecting lines $g \neq h$, find vectors A to C so that g and h have equations $X = At + C$ and $X = Bt + C$. Then $\det[ABX] = \det[ABC]$ is a linear equation for the plane through g and h .

Exercise 4.10.6, part 1. The plane with parametric equations $X = At + Bu + C$ has linear equation $\det[ABX] = \det[ABC]$. ♦

Exercise 34. Extend as follows the tool kit that you revised in exercise 33:

Task:

Given

Determine or find

1. lines g and h , ... their angle;
2. planes δ and ε , ... their dihedral angle;
3. points P, Q not on plane ε , . . . whether P and Q lie on the same side of ε ;
4. points P, Q coplanar with whether P and Q lie on the same side of g in their plane;
a line g but not on it,
5. noncollinear points P, Q, R , . . . area of ΔPQR ;
6. noncoplanar points P, Q, R, S , . . volume of tetrahedron $PQRS$.

Solution, Task 1. The angle between lines with parametric equations $X = At + C$ and $X = Bt + D$ has the same measure as that between A and B , namely

$$\arcsin \frac{\|A \times B\|}{\|A\| \|B\|}.$$

Task 2. The dihedral angle between planes with linear equations $X \cdot A = c$ and $X \cdot B = c$ has the same measure as the angle between A and B , which is the solution to task 1.

Task 3. Points P and Q lie on the same side of plane ABC just in case $\det[P' A' B' C']$ and $\det[Q' A' B' C']$ have the same sign, where columns P' to C' are formed from P to C by appending entries -1 .

Task 4. Points P and Q coplanar with line AB lie on the same side of it just in case $\det[PAB]$ and $\det[QAB]$ have the same sign.

Task 5. $a\Delta PQR = \frac{1}{2} \|(P - R) \times (Q - R)\|$.

Task 6. $vPQRS = \frac{1}{6} |\det[P' Q' R' S']|$, where columns P' to S' are formed from P to S by appending entries -1 . ♦

The next two exercises continue the program begun in exercise 2 to prove the major theorems of this chapter analytically. Desargues' theorem is harder to handle than Menelaus' or Ceva's, even though it doesn't involve directed distances. Thus it's placed by itself. The suggested analytic methods don't

favor any particular choice of coordinate system; their heavy reliance on parametric equations and vector algebra makes that nearly irrelevant.

Exercise 35. Prove Menelaus' and Ceva's theorems analytically. First, show that if $\vec{XY}$ is a directed line, $T = tX + (1 - t)Y$, and $t \neq 0$, then

$$\frac{X \text{ to } T}{T \text{ to } Y} = \frac{1 - t}{t}.$$

Use fractions like this to formulate Menelaus' product, and use the determinant conditions in section 5.6 to indicate when three points are collinear or three lines have a common point.

Solution. Use the coordinate system in figure 5.11.2. Verify the equation for $(X \text{ to } T)/(T \text{ to } Y)$ by checking the signs of its numerator and denominator. Now assign scales to $\vec{BC}$, $\vec{CA}$, $\vec{AB}$ in order to use directed distances, and select points $S, T, U \neq A, B, C$ on them so that

$$\begin{aligned} S &= sB + (1 - s)C \\ T &= tC + (1 - t)A \\ U &= uA + (1 - u)B \end{aligned} \quad \frac{(A \text{ to } U)(B \text{ to } S)(C \text{ to } T)}{(U \text{ to } B)(S \text{ to } C)(T \text{ to } A)} = \frac{1 - s}{s} \frac{1 - t}{t} \frac{1 - u}{u}$$

for some $s, t, u \neq 0$. Then Menelaus' product, just displayed on the right, equals -1 if and only if $s + t + u - st - tu - us = 1$; and S, T , and U fall on a line just in case

$$\begin{aligned} 0 &= \det [S - U, T - U] \\ &= \det \begin{bmatrix} sx_B - ux_A - (1 - u)x_B & (1 - t)x_A - ux_A - (1 - u)x_B \\ (1 - s)y_C & ty_C \end{bmatrix} \\ &= (x_A - x_B)y_C(s + t + u - st - tu - us - 1). \end{aligned}$$

That yields Menelaus' theorem.

Lines $\vec{AS}$, $\vec{BT}$, and $\vec{CU}$ have x, y equations

$$\begin{aligned} y_C(1 - s)x + (x_A - x_Bs)y - y_Cx_A(1 - s) &= 0 \\ y_Ctx + (x_B - x_A + x_At)y - x_By_Ct &= 0 \\ y_Cx + (ux_A - ux_B + x_B)y - y_C(ux_A - ux_B + x_B) &= 0, \end{aligned}$$

so they're copencilar just in case

$$\begin{aligned} 0 &= \det \begin{bmatrix} y_C(1 - s) & x_A - x_Bs & -y_Cx_A(1 - s) \\ y_Ct & x_B - x_A + x_At & -x_By_Ct \\ y_C & ux_A - ux_B + x_B & -y_C(ux_A - ux_B + x_B) \end{bmatrix} \\ &= -y_C^2(x_A - x_B)^2(1 - s - t - u + st + tu + us - 2stu). \end{aligned}$$

But Menelaus' product equals $+1$ just in case

$$1 - s - t - u + st + tu + us - 2stu = 0.$$

That yields Ceva's theorem. ♦

Exercise 36. Prove Desargues' theorem analytically.

Solution. The concurrent case in figure 5.3.1 is proved here. You can provide a proof for the parallel case. Choose an x, y coordinate system with origin O . Then

$$A' = uA \quad B' = vB \quad C' = wC$$

for some nonzero scalars u, v, w . Next,

$$\begin{aligned} L &= B + x(C - B) = B' + x'(C' - B') \\ M &= C + y(A - C) = C' + y'(A' - C') \\ N &= A + z(B - A) = A' + z'(B' - A') \end{aligned}$$

for some scalars x, y, z, x', y', z' . The preceding nine equations yield

$$\begin{aligned} L &= (1 - x)B + xC = v(1 - x')B + wx'C \\ M &= (1 - y)C + yA = w(1 - y')C + uy'A \\ N &= (1 - z)A + zB = u(1 - z')A + vz'B. \end{aligned}$$

This L equation entails

$$[(1 - x) - v(1 - x')]B + [x - wx']C = 0.$$

If one of these coefficients of B and C were nonzero, then one of B and C would be a scalar multiple of the other, so B and C would be collinear with O , which isn't the case. You can argue similarly with the M and N equations, and get

$$\begin{aligned} 1 - x &= v(1 - x') & x &= wx' \\ 1 - y &= w(1 - y') & y &= uy' \\ 1 - z &= u(1 - z') & z &= vz'. \end{aligned}$$

These six equations yield

$$x = w \frac{v - 1}{v - w} \quad 1 - x = v \frac{1 - w}{v - w}$$

$$y = u \frac{w - 1}{w - u} \quad 1 - y = w \frac{1 - u}{w - u}$$

$$z = v \frac{u - 1}{u - v} \quad 1 - z = u \frac{1 - v}{u - v}$$

$$L = (1 - x)B + xC = v \frac{1 - w}{v - w} B + w \frac{v - 1}{v - w} C$$

$$M = (1 - y)C + yA = w \frac{1 - u}{w - u} C + u \frac{w - 1}{w - u} A$$

$$N = (1 - z)A + zB = u \frac{1 - v}{u - v} A + v \frac{u - 1}{u - v} B$$

$$(v - w)L = v(1 - w)B + w(v - 1)C$$

$$(w - u)M = w(1 - u)C + u(w - 1)A$$

$$(u - v)N = u(1 - v)A + v(u - 1)B$$

$$\begin{aligned} (1 - u)(v - w)L - (v - 1)(w - u)M \\ = (1 - u)v(1 - w)B - (v - 1)u(w - 1)A \\ = -(1 - w)(u - v)N \end{aligned}$$

$$\begin{aligned} M &= \frac{(1 - u)(v - w)}{(v - 1)(w - u)} L + \frac{(1 - w)(u - v)}{(v - 1)(w - u)} N \\ \frac{(1 - u)(v - w)}{(v - 1)(w - u)} + \frac{(1 - w)(u - v)}{(v - 1)(w - u)} &= 1. \end{aligned}$$

Thus L , M , and N are collinear. (If $v = 1$, then $B = B' = L = N$ and the result still holds. If $w = u$, then $\vec{CA} \parallel \vec{C'A'}$, which is not so.) ♦

In exercise 2, you worked through analytic proofs of the edge bisectors, altitudes, and medians theorems, but not the angle bisectors theorem. It doesn't yield easily to analytic methods, mainly because the formulas for the slopes of the bisectors are too complicated. The problem is posed here, with the suggestion that you use algebra software to carry out, or at least verify, the tedious calculations. For this work and exercise 38, the author used the *weakest* such package available at the time.⁸ It was able to carry out the computations, but only barely. Learning how to control it, though, led to a deeper understanding of the algebra. You have to know algebra very well in order to use algebra software effectively.

Exercise 37. Prove the angle bisectors theorem analytically, by the method of exercise 2.

Solution. Use the coordinate system in figure 5.11.2. First,

$$\begin{aligned} \cos m\angle A &= -\frac{x_A}{\sqrt{x_A^2 + y_C^2}} & \sin m\angle A &= \frac{y_C}{\sqrt{x_A^2 + y_C^2}} \\ \cos m\angle B &= \frac{x_B}{\sqrt{x_B^2 + y_C^2}} & \sin m\angle B &= \frac{y_C}{\sqrt{x_B^2 + y_C^2}}. \end{aligned}$$

⁸ Derive 1989-.

Next, the slopes of the $\angle A$ and $\angle B$ bisectors are

$$\begin{aligned}\tan \frac{1}{2} m\angle A &= \frac{1 - \cos m\angle A}{\sin m\angle A} = \frac{x_A + \sqrt{x_A^2 + y_C^2}}{y_C} \\ -\tan \frac{1}{2} m\angle B &= -\frac{1 - \cos m\angle B}{\sin m\angle B} = -\frac{\sqrt{x_B^2 + y_C^2} - x_B}{y_C}.\end{aligned}$$

Thus the $\angle A$ and $\angle B$ bisectors have equations

$$y = \frac{x_A + \sqrt{x_A^2 + y_C^2}}{y_C} (x - x_A) \quad y = \frac{x_B - \sqrt{x_B^2 + y_C^2}}{y_C} (x - x_B).$$

Equating their right-hand sides, you can solve for

$$x = \frac{x_A^2 - x_B^2 + x_A \sqrt{x_A^2 + y_C^2} + x_B \sqrt{x_B^2 + y_C^2}}{x_A - x_B + \sqrt{x_A^2 + y_C^2} + \sqrt{x_B^2 + y_C^2}}.$$

Multiply the top and bottom of this fraction by

$$x_A - x_B - \sqrt{x_A^2 + y_C^2} - \sqrt{x_B^2 + y_C^2}.$$

Expanding and simplifying the bottom, you get

$$-2x_A x_B - 2y_C^2 - 2\sqrt{x_A^2 + y_C^2} \sqrt{x_B^2 + y_C^2}.$$

Now multiply the top and bottom by

$$-2x_A x_B - 2y_C^2 + 2\sqrt{x_A^2 + y_C^2} \sqrt{x_B^2 + y_C^2}.$$

Expanding and simplifying the top and bottom, you get

$$\begin{aligned}x &= \frac{-2y_C^2(x_A - x_B)^2 \left(x_A + x_B + \sqrt{x_A^2 + y_C^2} - \sqrt{x_B^2 + y_C^2} \right)}{-4y_C^2(x_A - x_B)^2} \\ &= \frac{x_A + x_B + \sqrt{x_A^2 + y_C^2} - \sqrt{x_B^2 + y_C^2}}{2}.\end{aligned}$$

The $\angle A$ and $\angle B$ bisectors meet at point $I = \langle x, y \rangle$. Next, use vectors

$$\begin{aligned}\vec{CA} &= A - C = \langle x_A, -y_C \rangle \\ \vec{CI} &= I - C = \langle x, y - y_C \rangle \\ \vec{CB} &= B - C = \langle x_B, -y_C \rangle\end{aligned}$$

to determine

$$\cos m\angle ACI = \frac{x_A x - y_C(y - y_C)}{\sqrt{x_A^2 + y_C^2} \sqrt{x^2 + (y - y_C)^2}}$$

$$\cos m\angle BCI = \frac{x_B x - y_C(y - y_C)}{\sqrt{x_B^2 + y_C^2} \sqrt{x^2 + (y - y_C)^2}}.$$

To show that I lies on the $\angle C$ bisector, you must verify that

$$\frac{x_A x - y_C(y - y_C)}{\sqrt{x_A^2 + y_C^2}} = \frac{x_B x - y_C(y - y_C)}{\sqrt{x_B^2 + y_C^2}}.$$

Substitute for the left- and right-hand occurrences of y the right-hand sides of the equations for the $\angle A$ and $\angle B$ bisectors, expand, and simplify each side of the resulting equation separately. You'll see that the desired equation is equivalent to

$$\sqrt{x_A^2 + y_C^2} - (x - x_A) = \sqrt{x_B^2 + y_C^2} + (x - x_B).$$

But that's in turn equivalent to

$$x = \frac{x_A + x_B + \sqrt{x_A^2 + y_C^2} - \sqrt{x_B^2 + y_C^2}}{2},$$

which was verified earlier. ♦

Exercise 38. This final exercise leads you through an analytic proof that the incircle of $\triangle ABC$ is internally tangent to the Feuerbach circle. It proceeds somewhat like Feuerbach's published proof (1822). But you're encouraged to employ algebra software to confront the calculations directly, instead of Feuerbach's devices to limit their complexity.

Suggestion: Use the coordinate system in figure 5.11.2, with

$$\begin{array}{lll} A = \langle x_A, 0 \rangle & B = \langle x_B, 0 \rangle & C = \langle 0, y_C \rangle \\ x_A < 0 & x_B > 0 & y_C > 0. \end{array}$$

Use the standard abbreviations a for $BC = \sqrt{x_B^2 + y_C^2}$ and b for $CA = \sqrt{y_C^2 + x_A^2}$ in your computations, and replace a^2 and b^2 by $x_B^2 + y_C^2$ and $y_C^2 + x_A^2$ as appropriate. By introducing variables a and b in addition to x_A , x_B , and y_C , you minimize the role of square roots. You can apply polynomial algebra, and your polynomials will be linear in a and b .

First, compute the inradius r of $\triangle ABC$ and the coordinates of its incenter I as in exercises 1, 2, and 37. Second, find the circumradius R of $\triangle ABC$ via one of the equations derived in this chapter. Since the medial triangle $\triangle A'B'C'$ is similar to $\triangle ABC$ with ratio $1/2$, its circumradius $R' = 1/2 R$.

Third, compute the coordinates of the circumcenter O' of the medial triangle. Fourth, show that Feuerbach's result is implied by the equation $(IO')^2 = (r - R')^2$. Finally, rewrite this equation as $p(x_A, x_B, y_C, a, b) = 0$, where p is a sixth-degree homogeneous polynomial. You may then use algebra software to compute the coefficients of p —the author's polynomial had 168 terms—and verify that they're all zero.

Solution. To find the inradius r of ΔABC , use the fact that its area is r times its semiperimeter:

$$\begin{aligned} \frac{1}{2}(x_B - x_A)y_C &= r \cdot \frac{1}{2}(x_B - x_A + a + b) \\ r &= \frac{(x_B - x_A)y_C}{x_B - x_A + a + b}. \end{aligned}$$

The incenter $I = \langle x_I, y_I \rangle$ is the point on the $\angle A$ bisector with $y_I = r$. To find x_I , first calculate the bisector's slope

$$\tan \frac{1}{2} m\angle A = \frac{1 - \cos m\angle A}{\sin m\angle A} = \frac{1 + \frac{x_A}{b}}{\frac{y_C}{b}} = \frac{x_A + b}{y_C}.$$

Thus the equation is

$$y = \frac{x_A + b}{y_C}(x - x_A)$$

and you can set $y = r$ and solve for $x = x_I$:

$$x_I = x_A + \frac{y_C r}{x_A + b}.$$

According to the law of sines, the ratio of the length of an edge of ΔABC to the sine of the measure of the opposite angle is its circumdiameter $2R$. Therefore,

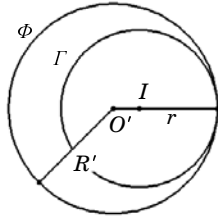
$$R' = \frac{1}{2}R = \frac{1}{4}(2R) = \frac{a}{4 \sin m\angle A} = \frac{a}{4 \frac{y_C}{b}} = a \frac{b}{4 y_C}.$$

The edge midpoints of ΔABC are

$$\begin{aligned} A' &= \langle \frac{1}{2}x_B, \frac{1}{2}y_C \rangle \\ B' &= \langle \frac{1}{2}x_A, \frac{1}{2}y_C \rangle \\ C' &= \langle \frac{1}{2}(x_A + x_B), 0 \rangle. \end{aligned}$$

The perpendicular bisector of edge $\overline{A'B'}$ of the medial triangle has equation

$$x = \frac{1}{2}(\frac{1}{2}x_A + \frac{1}{2}x_B) = \frac{1}{4}(x_A + x_B)$$

**Figure 5.11S.8**

Internally tangent circles Γ, Φ with centers I, O' and radii r, R' , where $r < R'$ and $IO' = R' - r$.

and its circumcenter $O' = \langle x_{O'}, y_{O'} \rangle$ is the point on this line equidistant from B' and C' . To find $y_{O'}$, solve the equation

$$[x_{O'} - \frac{1}{2}x_A]^2 + [y_{O'} - \frac{1}{2}y_C]^2 = [x_{O'} - \frac{1}{2}(x_A + x_B)]^2 + [y_{O'} - 0]^2,$$

and get

$$x_{O'} = \frac{x_A + x_B}{4} \quad y_{O'} = \frac{y_C^2 - x_A x_B}{4 y_C}.$$

To prove that circles Γ and Φ with centers I and O' and radii r and R' are internally tangent, as in figure 5.11S.8, it's clearly sufficient to show that $IO' = |r - R'|$, which is equivalent to $(IO')^2 = (r - R')^2$.

Since $y_I = r$, this amounts to

$$\begin{aligned} (x_I - x_{O'})^2 + (r - y_{O'})^2 &= (r - R')^2 \\ (x_I - x_{O'})^2 - 2r y_{O'} + y_{O'}^2 &= -2r R' + R'^2 \\ (x_I - x_{O'})^2 &= 2r(y_{O'} - R') - (y_{O'}^2 - R'^2) \\ (x_I - x_{O'})^2 &= (2r - y_{O'} - R')(y_{O'} - R'), \end{aligned}$$

where $x_I, x_{O'}, y_{O'}, r$ and R' are given by earlier formulas.

Using algebra software, compute

$$\begin{aligned} (x_I - x_{O'})^2 &= \frac{36x_A^3 x_B a b + \cdots + 10x_B^2 y_C^4}{16(x_B - x_A + a + b)^2 (x_A + b)^2} = \lambda, \\ (2r - y_{O'} - R')(y_{O'} - R') &= \frac{8x_A a b + \cdots + 8x_B y_C^2}{16(x_B - x_A + a + b)} = \rho. \end{aligned}$$

The numerators are homogeneous polynomials in x_A, x_B, y_C, a , and b with 31 and 14 terms. The desired equation $\lambda = \rho$ holds just in case

$$(\text{Numerator of } \lambda) - (x_B - x_A + a + b)(x_A + b)^2 (\text{Numerator of } \rho) = 0.$$

This polynomial has $4 \times 3 \times 14 = 168$ terms. Use algebra software to verify that it's zero. ♦

You may want to continue exercise 38 as a project: Prove that the excircles are externally tangent to the Feuerbach circle, and investigate the devices Feuerbach used to manage the algebra.

6.11

Solutions to exercises on plane isometries and similarities

Concepts

- Curved mirrors
- Classifying isometries defined as compositions or by equations
- Constructions with isometries and similarities
- Billiards and dual billiards problems
- Equations of reflections
- Determining equations of isometries and similarities
- Coordinate transformations
- Alias* and *alibi* interpretations of equations
- Reflection calculus
- Distance-preserving functions
- Dilatations

This section contains forty-five exercises related to earlier parts of the chapter. Like other exercises in this book, they're essential components of its development. A few—for example, exercises 1, 5, and 12—provide more experience

with reasoning techniques introduced earlier. Others, particularly those on the classification and the equations of isometries, show you how to do the computations required to use the theorems derived in the text. The theory is streamlined into an efficient outline of the results important for applications. But those computations seem unwieldy at first. That would obscure the main points of the theory, so the computations are clustered here. Some may require awkward matrix arithmetic. There was no effort to avoid that. If you find a calculation too tedious, use mathematical software.¹

This chapter's theory is often applied to analytic geometry problems. That work usually has a different flavor because it uses *coordinate* transformations. Those apply to the numbers used to locate points, not to the points themselves, like geometric transformations here. Exercise 31 shows how to make the transition. A few exercises introduce topics that could be developed much further. Those on reflection calculus, and on dilatations, for example, lead into deep studies in foundations of geometry. Chapter 7 extends the theory of isometries and similarities to three dimensions. Most of the exercises in this section can be extended as well. You'll see that in section 7.7.

Basic exercises

Exercises 1 and 2 present some important properties that apply to transformations in general, not just those in geometry. For instance, exercise 2, part 1, asks what you can say about functions φ and χ whose composition is surjective, and what you can *not* say, in general. To show that you can *not* claim that φ or χ has some property $\mathcal{P}$, you must provide a *counterexample*: specific functions whose composition is surjective, but for which $\mathcal{P}$ is not true. Searching for counterexamples for exercise 2, you should concentrate on functions whose domains and ranges are *very small* finite sets. You won't need more than that to make the required properties fail, and you'll avoid confusion by other phenomena that are really irrelevant. That kind of reasoning may seem nongeometric to you, but no—those tiny counterexamples are always contained in the larger ones that may seem closer to geometry, and they're much easier to understand.

Exercise 1. Prove that two self-inverse transformations of a set D commute if and only if their composition is self-inverse.

Solution. Consider self-inverse transformations φ and ψ . If $\varphi\psi = \psi\varphi$, then $(\varphi\psi)(\varphi\psi) = (\varphi\psi)(\psi\varphi) = \varphi(\psi\psi)\varphi = \varphi\iota_D\varphi = \varphi\varphi = \iota_D$, so $\varphi\psi$ is self-inverse. If $\varphi\psi$ is self-inverse, then $\varphi\psi\varphi\psi = \iota_D$, hence $\varphi\psi\varphi = \psi^{-1} = \psi$, hence $\varphi\psi = \psi\varphi^{-1} = \psi\varphi$. ♦

¹ $X(\text{Plore})$ (Meredith 1993) is a good choice.

$\det AM = \det I = 1$, so $\det A \neq 0$, hence A is invertible. Thus (d) implies (a), so they're equivalent, too. ♦

Exercise 4. Let $n = 2$ or 3 . Show that the range of a linear function $X \rightarrow AX + B$ that maps $\mathbb{R}^n$ to itself is a point, line, plane, or the entire space. In which cases is φ bijective? *Suggestion:* First consider the function $\varphi : X \rightarrow AX$.

Solution. For $k = 1$ to n let I_k be the point with coordinate 1 on the k th axis. Then $AX = A(x_1 I_1 + \cdots + x_n I_n) = x_1 AI_1 + \cdots + x_n AI_n$ for each point $X = \langle x_1, \dots, x_n \rangle$. If $AI_1 = \cdots = AI_n = O$, then $AX = O$ for all X , and the range of φ consists of O alone.

Suppose $AI_k \neq O$ for some k . If $AI_1, \dots, AI_n$ all fall on a line g through O , then $x_1 AI_1, \dots, x_n AI_n$ all fall on g , hence so does AX . Conversely, if P is a point on g , then $P = tAI_k = A(tI_k)$ for some scalar t , so P lies in the range of φ . That is, the range is exactly the set g .

Suppose O, AI_j , and AI_k are noncollinear for some j and k . If $AI_1, \dots, AI_n$ all fall on a plane ε through O , then $x_1 AI_1, \dots, x_n AI_n$ all fall on ε , hence so does AX . Conversely, if P is a point on ε , then $P = tAI_j + uAI_k = A(tI_j + uI_k)$ for some scalars t and u , so P lies in the range of φ . That is, the range is exactly the set ε (the entire plane if $n = 2$).

Suppose $O, AI_1, \dots, AI_n$ are noncoplanar (so $n = 3$). If P is any point, then $P = t_1 AI_1 + t_2 AI_2 + t_3 AI_3 = A(t_1 I_1 + t_2 I_2 + t_3 I_3)$ for some scalars t_1, t_2 , and t_3 , so P lies in the range of φ . That is, the range of φ is the entire space.

The range of $X \rightarrow AX + B$ consists of all vectors of the form $P + B$, where P lies in the range of φ . So it's the point B , a line, a plane (the entire plane if $n = 2$), or the entire space.

According to exercise 3, φ is bijective when $n = 2$ and its range is the entire plane, or when $n = 3$ and its range is the entire space. ♦

Theorem 6.6.3 shows that you can map any triangle onto any congruent one with an isometry; the uniqueness theorem, corollary 6.2.8, says that you can do that in only one way. The latter statement requires you to regard a triangle as an *ordered* triple of vertices. You'd lose the uniqueness if you defined a triangle as a mere *set* of vertices or as its boundary—the union of its edges—for there are several isometries that map the set of vertices and the boundary of an equilateral triangle onto themselves. You'll see much more detail on these considerations in chapter 8. In exercise 5 you can study the same question for a simpler example, a segment. As defined in section 3.2, a segment is the *set* of points between its vertices, *not* an ordered pair of vertices.

Exercise 5. Show that if Σ and Υ are segments of the same length, then there's a rotation ρ such that $\rho[\Sigma] = \Upsilon$. Could there be *another* isometry φ such that $\varphi[\Sigma] = \Upsilon$? Could there be another *rotation*?

Solution. Let $\Sigma = \overline{AB}$ and $\Upsilon = \overline{CD}$. Let g be the line containing Υ , O be its midpoint, and h be its perpendicular bisector. By lemma 6.6.2 there's an isometry χ such that $C = \chi(A)$ and $D = \chi(B)$; moreover $\chi = \iota$, or $\chi = \sigma_k$ for some line k , or χ is a rotation, or χ is a translation. In the first and third cases, let $\rho = \chi$. In the second, let $l = g$ or h , whichever isn't parallel to k , and let $\rho = \sigma_l \sigma_k$. In the last case, $ABDC$ is a parallelogram; let ρ be the half turn about its center.

If ρ is a rotation that maps Σ onto Υ , then $\sigma_g \rho$, $\sigma_h \rho$, and $\sigma_O \rho$ are three more isometries that map Σ onto Υ , and they're all distinct. That's all there are. For if an isometry φ maps Σ onto Υ , then $\varphi \rho^{-1}$ maps Υ onto itself, so it can only be ι , σ_g , σ_h , or σ_O . Unless ρ is a half turn, $\sigma_O \rho$ is a rotation. ♦

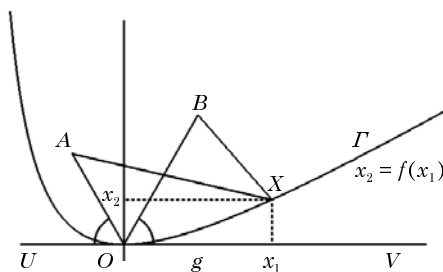
At the beginning of section 6.3 you saw how to find the shortest path from one point to another via a point on a mirror line g by reflecting one of the points across g . And you saw how this relationship corresponds to the equality of incidence and reflection angles. Exercise 6 shows that the relationship holds for a smoothly curved mirror as well. You need differential calculus to work with the curve.

Exercise 6. Consider a smooth curve Γ and points A and B not on Γ . (See figure 6.11.1.) Suppose point O lies on Γ and

$$AO + OB \leq AX + XB$$

for every point X on Γ . Let g be the tangent to Γ at O and select points U and V on g such that $U \cdot O \cdot V$. Prove that $\angle UOA \cong \angle VOB$. *Suggestion:* Use a coordinate system with first axis g and origin O . Since Γ is smooth, there's a differentiable function f on an open interval $\mathcal{I}$ containing 0 such that for any $x_1 \in \mathcal{I}$, a point $X = \langle x_1, x_2 \rangle$ lies on Γ just in case $x_2 = f(x_1)$.

Figure 6.11.1
Exercise 6



Solution. Let $A = \langle a_1, a_2 \rangle$, $B = \langle b_1, b_2 \rangle$, and consider the function

$$d(x_1) = AX + BX = \sqrt{(x_1 - a_1)^2 + (x_2 - a_2)^2} + \sqrt{(x_1 - b_1)^2 + (x_2 - b_2)^2}.$$

Then $0 = d'(0)$ since $d(x_1)$ is minimum when $x_1 = 0$. That yields

$$\begin{aligned} & \cos m\angle UOA - \cos m\angle VOB \\ &= \frac{-a_1}{\sqrt{(-a_1)^2 + (-a_2)^2}} + \frac{-b_1}{\sqrt{(-b_1)^2 + (-b_2)^2}} = 0. \diamond \end{aligned}$$

The next exercise invites you to use a technique from section 6.3 to derive Fagnano's theorem, a morsel of triangle geometry mentioned in section 5.8.

Exercise 7. Show that of all triangles with vertices X , Y , and Z on the edges of an acute $\triangle ABC$, the orthic triangle has the shortest perimeter. *Suggestion:* Show first that for a given X on $\overline{BC}$ the shortest $XY + YZ + ZX$ is $\sigma_{AB}(X) \sigma_{CA}(X) = 2(AX) \sin m\angle A$.

Solution. The perimeter $p(X) = ZX + XY + YZ$ is shortest when $\sigma_{CA}(X)$, $\sigma_{AB}(X)$, Y , and Z fall on a line. Then $p(X)$ is the base of isosceles triangle $\triangle \sigma_{AB}(X) A \sigma_{CA}(X)$, whose opposite angle has measure $2m\angle A$, and whose other edges have length AX . This implies $p(X) = 2(AX) \sin m\angle A$, which is minimized when X is closest to A —that is, when X is the foot of the altitude through A . Similarly, Y and Z must be the feet of the other altitudes.² $\diamond$

Classifying isometries

Most exercises under this heading ask you to classify an isometry. That means decide whether it's the identity or one of these types of isometries, and provide the information in the right-hand column:

<i>For a</i>	<i>provide</i>
reflection	axis
translation	vector
rotation	center and an angle parameter
glide reflection	axis and vector.

Exercises 8 and 9 are aperitifs—you'll use this classification to verify some properties of isometries in general.

² This solution is due to Leopold Fejér. See Rademacher and Toeplitz [1933] 1957, sections 5–6.

Exercise 8. Show that an isometry φ is even if and only if $\varphi = \chi\chi$ for some isometry χ .

Solution. Clearly, $\chi\chi$ is even. On the other hand, an even isometry φ is a translation with vector V or a rotation with angle parameter $\theta \neq 0^\circ$. In the first case, let χ be the translation with vector $\frac{1}{2}V$. In the second, let χ be the rotation about the same center with angle parameter $\frac{1}{2}\theta$. In either case, $\varphi = \chi\chi$. ♦

Exercise 9. Show that every plane isometry is the composition of two self-inverse isometries.

Solution. Rotations and translations are compositions of two line reflections. If φ is a glide reflection with axis g , then $\varphi = \sigma_g\tau$ for some translation τ with vector parallel to g . Therefore, $\tau = \sigma_O\sigma_P$ for some points O and P on g , so $\varphi = \sigma_g\sigma_O\sigma_P = \sigma_h\sigma_P$, where h is the perpendicular to g through O . ♦

Exercise 10. Prove a result like theorem 6.9.3 by continuing the proofs of corollaries 6.5.7 and 6.5.8 and analyzing the angles formed by the lines mentioned there, without referring to any equations of transformations. *Suggestion:* Be careful about the *signs* of the angle parameters. You don't need to get precisely the same result as the theorem. It was postponed from section 6.5 to section 6.9 because consistent assignment of angle parameters for different centers is so hard to describe.

Solution. Composing a rotation and a translation, consider corresponding angles formed by parallel lines and a transversal. For two rotations consider an exterior angle of a triangle. ♦

Exercise 11 presents a nearly obvious result that you may use in several later exercises to find the axis of a glide reflection.

Exercise 11. Show that the axis of a glide reflection $\varphi : X \rightarrow X'$ consists of the midpoints of all segments $\overline{XX'}$.

Solution. Clearly, all such midpoints lie on the axis. Conversely, if φ has vector V and axis g and M lies on g , then $X = M - \frac{1}{2}V$ lies on g , $X' = X + V$, and M is the midpoint of $\overline{XX'}$. ♦

Exercise 12. Construct a table analogous to the one following corollary 6.9.2, to show the possible sets of lines that are fixed by the various isometries. A fixed line might or might not consist entirely of fixed points. Lines that do are called *pointwise* fixed. What isometries have pointwise fixed lines? What isometries have fixed lines, *no* point of which is fixed?

Solution.

Class	Isometry	Fixed line set
(0)	ι	all lines
(1)	σ_g	$\{h : h = g \text{ or } h \perp g\}$
(2a)	τ_V with $V \neq O$	$\{h : h \parallel \overrightarrow{OV}\}$
(2b')	ρ_θ with $\theta \bmod 360^\circ \neq 0^\circ, 180^\circ$	empty
(2c)	σ_O	$\{h : O \text{ on } h\}$
(3)	$\sigma_g \tau_V$ with $V \neq O$	$\{g\}$

The fixed lines of class (0) and the fixed line g in Class (1) are the only pointwise fixed lines. All fixed lines except those of classes (2a) and (3) contain fixed points. ♦

The remaining exercises under this heading and some later ones require you to classify various isometries according to the scheme presented earlier. The main tools available are their fixpoints and parity—discussed in earlier sections—and exercises 11 and 12. *Use those tools!*

Exercise 13. Let $V_1 V_2 \cdots V_n$ be a convex polygon and $\theta_1, \theta_2, \dots, \theta_n$ be the measures of the corresponding vertex angles. At each vertex V_i draw arc $A_i B_i$ with measure θ_i and radius $V_i A_i = V_i B_i$ small enough that no two of these arcs intersect. (See figure 6.11.2 for an example with $n = 4$.) Consider translations

$$\begin{aligned} \tau_1 & \text{ with vector } \overrightarrow{B_1 A_2} \\ \tau_2 & \text{ with vector } \overrightarrow{B_2 A_3} \\ & \vdots \\ \tau_n & \text{ with vector } \overrightarrow{B_n A_1} \end{aligned}$$

and for $i = 1, \dots, n$ consider rotation ρ_i with center V_i and angle parameter θ_i . Classify the isometry $\tau_n \rho_n \tau_{n-1} \rho_{n-1} \cdots \tau_1 \rho_1$.³

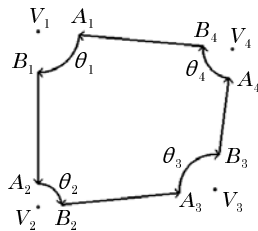


Figure 6.11.2
Exercise 13

³ This exercise is due to Lawrence Levy (1970, 32).

Solution. The composition φ is even and fixes A_1 , so it's a rotation about A_1 with angle parameter

$$\begin{aligned}\eta &= (\theta_1 + \cdots + \theta_{n-1} + \theta_n) \bmod 360^\circ \\ &= (180^\circ n - 360^\circ) \bmod 360^\circ = (180^\circ n) \bmod 360^\circ\end{aligned}$$

This is 0° if n is even, or 180° if it's odd. Thus $\varphi = \iota$ if n is even and $\varphi = \sigma_{A_1}$ if n is odd. ♦

Exercise 14. Classify the composition of glide reflections with perpendicular axes g and h and vectors V and W .

Solution. Lines g and h meet at a point O . Let $P = O + \frac{1}{2}V$, $Q = O - \frac{1}{2}W$, and $R = O + \frac{1}{2}V - \frac{1}{2}W$, so that $POQR$ is a rectangle and the glide reflections are $\varphi = \tau_V \sigma_g = \sigma_P \sigma_O \sigma_g$ and $\chi = \sigma_h \tau_W = \sigma_h \sigma_O \sigma_h$. Then

$$\varphi\chi = \sigma_P \sigma_O \sigma_g \sigma_h \sigma_O \sigma_Q = \sigma_P \sigma_O \sigma_O \sigma_O \sigma_Q = \sigma_P \sigma_O \sigma_Q = \sigma_{P-O+Q} = \sigma_R. \quad \blacklozenge$$

Exercise 15. Classify the compositions of the reflection across the first coordinate axis and the translation with vector $\langle 1, 2 \rangle$, in both orders.

Solution. Consider the first coordinate axis h , the origin O , the points $U = \langle 1, 0 \rangle$, $V = \langle 0, 1 \rangle$, and let $W = \langle 1, 2 \rangle$. By theorem 6.7.2, $\varphi = \tau_W \sigma_h$ is a glide reflection. Since $\varphi(O) = W$ and $\varphi(U) = U + W$, the points $\frac{1}{2}W$ and $\frac{1}{2}(U + U + W) = U + \frac{1}{2}W$ lie on its axis. The axis is therefore the line $x_2 = 1$. Since $\varphi(V) = -V + W$, φ has vector $W - 2V = U$. Similarly, $\chi = \sigma_h \tau_W$ is a glide reflection. Since $\chi(O) = U - 2V$ and $\chi(U) = 2U - 2V$, the points $\frac{1}{2}U - V$ and $\frac{3}{2}U - V$ lie on its axis. The axis is therefore the line $x_2 = -1$. Since $\chi(-V) = U - V$, χ has vector $U - V - (-V) = U$. ♦

Exercise 16. Let a , b , and c denote the lines containing the edges opposite vertices A , B , and C of $\triangle ABC$. Let D , E , and F denote the feet of the corresponding altitudes. Show that D and F lie on the axis of $\varphi = \sigma_a \sigma_b \sigma_c$. Classify φ when

- (a) $m\angle A = m\angle B = 60^\circ$ (c) $m\angle A = m\angle C = 45^\circ$
 (b) $m\angle A = m\angle B = 45^\circ$ (d) $m\angle A = 30^\circ$ and $m\angle C = 90^\circ$.

Solution. By theorem 6.7.2, $\varphi: X \rightarrow X'$ is a glide reflection. A lies on b and c , so $A' = \sigma_a \sigma_b \sigma_c(A) = \sigma_a \sigma_b(A) = \sigma_a(A)$, and by exercise 11, the midpoint D of $\overline{AA'}$ lies on the axis g . Moreover, $\varphi^{-1} = \sigma_c \sigma_b \sigma_a$ is a glide reflection with the same axis, so by the same argument, F lies on g . This result describes the axes for examples (a) to (d). Their vectors are $\vec{FF'}$ =

- (a) $1.5\vec{AC}$ (c) $\vec{AC}$
 (b) $2\vec{FC}$ (d) $2\vec{FC}$. ♦

Constructions

Following a more traditional approach to geometry, you'd have paid more attention to constructing the objects under study—probably with tools limited to straightedge and compasses. In fact, Euclid's axiomatization is sometimes viewed as a study of just those constructions, rather than of the space in which we live. This text has avoided them because they're no longer in the mathematical mainstream. The next seven exercises, however, merge transformational geometry into the older stream.⁴ Each can be solved by traditional methods with no reference to transformations, but the isometries and similarities you've studied in this chapter shed more light on them. And if you seek the *simplest* constructions with straightedge and compasses, transformations may point the way.

Exercise 17. Suppose you want to carom ball A off the back, right, and front walls g , h , and k of a billiards table, to strike ball B , as in figure 6.11.3. How do you find the point P on g at which you must aim? *Suggestion:* Try more modest coups first.

Solution. Find $B_1 = \sigma_k(B)$, $B_2 = \sigma_h \sigma_k(B)$, and $B_3 = \sigma_g \sigma_h \sigma_k(B)$, and let P be the intersection of g with $\overrightarrow{AB_3}$. First, A will strike g at P , and rebound along $\sigma_g[\overrightarrow{AB_3}] = \overrightarrow{PB_2}$. Next, it will strike h at a point Q and rebound along $\sigma_h[\overrightarrow{PB_2}] = \overrightarrow{QB_1}$. Then it will strike k at a point R and rebound along $\sigma_k[\overrightarrow{QB_1}] = \overrightarrow{RB}$. Therefore it will strike B . ♦

Exercise 18. Consider nonparallel lines $g \neq h$ and three points P , Q , and R . Find a three-segment path beginning on g and ending on h , such that P , Q , and R are the midpoints of its segments. Can there be more than one such path? What if $g = h$ or $g \parallel h$?

Solution. Let S be the ending point on h . The start of the third segment, and the end of the second, is $\sigma_R(S)$, which must lie on $\sigma_R[h]$. The start of the second segment, and the end of the first, is $\sigma_Q \sigma_R(S)$, which must

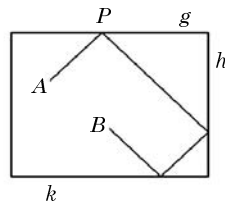


Figure 6.11.3
Exercise 17

⁴ Several exercises under this heading are due to Max Jeger [1964] 1966.

lie on $\sigma_Q\sigma_R[h]$. The start of the first segment is $O = \sigma_P\sigma_Q\sigma_R(S)$, which must lie on $k = \sigma_P\sigma_Q\sigma_R[h]$. Thus O must lie on $k \cap g$. Since $k \parallel h$, there's exactly one such path unless $g = h$ or $g \parallel h$. In this exceptional case there's no path if $k \parallel h$; if $k = h$, you can choose any point O on g . ♦

Did your solution of exercise 17 require that the table be rectangular? Probably not. Could you extend it to apply to a convex polygonal table with any number of edges? Probably so—you can buy or build novelty billiards games like that. Exercise 18 didn't place any limitations on points P , Q , and R . Can you extend it to apply to any number of points? Exercises 34 and 35 consider *closed* polygonal paths of this sort.

Exercise 19. Consider two circles on the same side of line g . Find a point P on g such that tangents through P to the circles make equal angles with g . How many such points are there? *Suggestion:* Reflect one of the circles across g , as in figure 6.11.4.

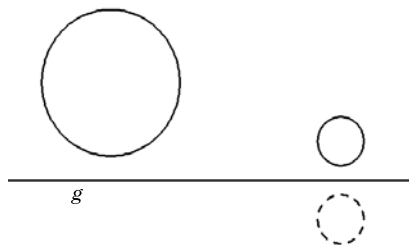
Solution. Call one circle Γ , the other Δ , and let $\Delta' = \sigma_g[\Delta]$. P can be the intersection of g with any of the four common tangents to Γ and Δ' . ♦

Exercise 20. Consider two points A and B on the same side of line g . Find points O , P , and Q on g such that $2\angle OPA = \angle QPB$. *Suggestion:* Consider exercise 19.

Solution. Let Γ be the circle with center A tangent to g , $B' = \sigma_g(B)$, and h be the tangent to Γ through B' that lies on the same side of AB' as B (either tangent if B , B' and A are collinear). Let P be the intersection of g and h . ♦

Exercise 21. Consider a point P and concentric circles Γ and Δ . Look for points Q and R on Γ and Δ such that ΔPQR is equilateral. Under what condition on P , Γ , and Δ is a solution possible? How many solutions can there be? *Suggestion:* On which *two* circles must Q lie?

Figure 6.11.4
Exercise 19



Solution. $R = \rho(Q)$ or $\rho^{-1}(Q)$, where ρ is a rotation about P with angle parameter 60° . Thus R lies on Δ and on $\rho[\Gamma]$ or $\rho^{-1}[\Gamma]$. Each of the latter circles could intersect Δ twice, once, or not at all. The number of intersections is the number of solutions. ♦

Exercise 22. Given $\triangle ABC$ with $m\angle A < 90^\circ$, find points C_1 and C_2 on the edge opposite C and points A_1 and B_1 on the edges opposite A and B such that $A_1B_1C_1C_2$ is a rectangle and $C_1C_2 = 2(B_1C_1)$.

Solution. Select any point B'_1 interior to the edge opposite B , and let C'_1 be the foot of the perpendicular from B'_1 to the edge line c opposite C . Find the point C'_2 on c such that $A-C'_1-C'_2$ and $C'_1C'_2 = 2(B'_1C'_1)$. Find the point A'_1 such that $A'_1B'_1C'_1C'_2$ is a rectangle. Let g be the line through A and A'_1 . Find the intersection A_1 of g with the edge opposite A . Pass $h \parallel c$ through A_1 . Find the intersection B_1 of h with the edge opposite B . Finally, find the feet C_1 and C_2 of the perpendiculars to c through B_1 and A_1 . ♦

Exercise 23. Consider a point B , two circles Γ and Δ , and any number $t > 0$. Find points C and D on Γ and Δ such that $C-B-D$ and $BC/BD = t$. How many solutions could there be? Describe, as completely as you can, the set $\mathcal{B}$ of points B for which a solution is possible. *Suggestions:* On which two circles must C lie? Describing $\mathcal{B}$ is difficult; you may want to plot the points B corresponding to several thousand pairs of points C and D on Γ and Δ . The details of your description may vary depending on the situation of the circles.

Solution. Let φ be the dilation or contraction with center B and ratio t . Then $C = \varphi\sigma_B(D)$, so C must lie on both Γ and $\varphi\sigma_B[\Delta]$. Those circles intersect twice, once, or not at all; the number of intersections is the number of solutions. Suppose Γ and Δ have centers O and P ; let $Q = O(1-t) + Pt$. If Γ and Δ are mutually exterior, then $\mathcal{B}$ is an annulus with center Q . Its inner boundary is tangent to the two common internal tangents to Γ and Δ ; its outer boundary is tangent to their two common external tangents. (These tangents are depicted in figure 4.6.4.) I don't know what $\mathcal{B}$ is when Γ and Δ aren't mutually exterior. ♦

Exercises on equations

Each exercise under this heading is concerned with the equation $X' = AX + B$ of a transformation $\varphi: X \rightarrow X'$. In exercises 24 to 26 the problem is simply to find the equation—that is, the entries of A and B .

Exercise 24. Two rotations about the point $Z = \langle 1, 2 \rangle$ each map the line $2x_1 - x_2 = 0$ onto the line $x_1 - x_2 = -1$. Determine their equations, accurate to four decimal places.

Solution. The lines' slopes are 2 and 1, so the angles from the first to the second are $\theta_1 = \frac{1}{4}\pi - \arctan 2 \approx -18.4349^\circ$ and $\theta_2 = \theta_1 + 180^\circ \approx 161.5651^\circ$. Each rotation has equation $X \rightarrow X' = A_\theta X + B$ for one value of θ and some vector B . Since they fix the center, $Z = A_\theta Z + B$, so $B = Z - A_\theta Z$. For θ_1 the rotation has equation

$$X' = \begin{bmatrix} 0.9487 & -0.3162 \\ 0.3162 & 0.9487 \end{bmatrix} X + \begin{bmatrix} 0.6838 \\ -0.2136 \end{bmatrix}$$

while for θ_2 it's

$$X' = \begin{bmatrix} -0.9487 & 0.3162 \\ -0.3162 & -0.9487 \end{bmatrix} X + \begin{bmatrix} 1.3162 \\ 4.2136 \end{bmatrix}. \blacklozenge$$

Exercise 25. Consider a point $P = \langle p_1, p_2 \rangle \neq O$ and the line $g \perp \vec{OP}$ through O .

Part 1. Show that σ_g has matrix

$$A = I - \frac{2}{P^t P} K, \text{ where } K = PP^t = \begin{bmatrix} p_1^2 & p_1 p_2 \\ p_2 p_1 & p_2^2 \end{bmatrix}.$$

Part 2. Consider the line $h \perp \vec{OP}$ through a point Q . Show that $\sigma_h : X \rightarrow X'$ has equation

$$X' = AX + \frac{2}{P^t P} KQ.$$

Part 3. Find an equation for the reflection across the line h with equation $x_1 + 2x_2 = 3$.

Part 4. Find an equation for the glide reflection with axis h and vector $\langle 2, -1 \rangle$.

Solution, Part 1. The solution for this part is contained in that for part (2).

Part 2. $X' = X + sP$ for some scalar s . The midpoint $M = \frac{1}{2}(X + X') = X + \frac{1}{2}sP$ lies on h , so $0 = P^t(M - Q) = P^t X + \frac{1}{2}sP^t P - P^t Q$, so

$$s = -\frac{2}{P^t P} P^t(X - Q)$$

$$X' = X - \frac{2}{P^t P} (P^t(X - Q))P$$

$$\begin{aligned}
&= X - 2 \frac{P^t X}{P^t P} P + 2 \frac{P^t Q}{P^t P} P \\
&= X - \frac{2}{P^t P} \left([p_1 \ p_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} + 2 \frac{P^t Q}{P^t P} P \\
&= X - \frac{2}{P^t P} \begin{bmatrix} (p_1 x_1 + p_2 x_2) p_1 \\ (p_1 x_1 + p_2 x_2) p_2 \end{bmatrix} + 2 \frac{P^t Q}{P^t P} P \\
&= X - \frac{2}{P^t P} \begin{bmatrix} p_1^2 & p_1 p_2 \\ p_2 p_1 & p_2^2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 2 \frac{P^t Q}{P^t P} P \\
&= \left(I - \frac{2}{P^t P} K \right) X + \frac{2}{P^t P} KQ.
\end{aligned}$$

Part 3. h passes through $Q = \langle 1, 1 \rangle$ and $h \perp \vec{OP}$, where $P = \langle 1, 2 \rangle$. Thus $P^t P = 1 \cdot 1 + 2 \cdot 2 = 5$,

$$\begin{aligned}
K &= \begin{bmatrix} 1^2 & 1 \cdot 2 \\ 2 \cdot 1 & 2^2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \\
A = I - \frac{2}{5} K &= \begin{bmatrix} \frac{3}{5} & -\frac{4}{5} \\ -\frac{4}{5} & -\frac{3}{5} \end{bmatrix} \\
\frac{2}{5} KQ &= \frac{2}{5} \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{2}{5} \begin{bmatrix} 3 \\ 6 \end{bmatrix} \\
X' &= \begin{bmatrix} \frac{3}{5} & -\frac{4}{5} \\ -\frac{4}{5} & -\frac{3}{5} \end{bmatrix} X + \begin{bmatrix} \frac{6}{5} \\ \frac{12}{5} \end{bmatrix}.
\end{aligned}$$

Part 4. The glide reflection is the composition $\tau \sigma_h$ of the reflection in part (3) and the translation τ with vector $\langle 2, -1 \rangle$, so its equation is

$$X' = \begin{bmatrix} \frac{3}{5} & -\frac{4}{5} \\ -\frac{4}{5} & -\frac{3}{5} \end{bmatrix} X + \begin{bmatrix} \frac{6}{5} \\ \frac{12}{5} \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{3}{5} & -\frac{4}{5} \\ -\frac{4}{5} & -\frac{3}{5} \end{bmatrix} X + \begin{bmatrix} \frac{16}{5} \\ \frac{2}{5} \end{bmatrix}. \blacklozenge$$

Exercise 26. Consider points $A = \langle 1, 0 \rangle$, $B = \langle 2, 0 \rangle$, and $C = \langle 0, 2 \rangle$, and the feet D and F of the altitudes of $\triangle ABC$ through A and C . According to theorem 5.8.1, $\triangle ABC \sim \triangle DBF$.

Part 1. Find the equations of the similarity $\varphi : X \rightarrow X'$ that maps $\triangle ABC$ onto $\triangle DBF$.

Part 2. What's its ratio?

Part 3. Express φ as a composition of some or all of these: reflection, translation, rotation, glide reflection, dilation, and contraction.

Part 4. Find its fixpoint.

Suggestion: The entries of the matrix and vector for φ form the solution of a system of six linear equations in six unknowns. You might want to solve that for part (1), then work on (3). Or you could solve (3) first geometrically, then do (1).

Solution, part 1. φ has equation $X' = MX + V$ for some matrix M and vector V . Regard equations $D = MA + V$, $B = MB + V$, and $F = MC + V$ as a system of six linear equations in the entries of M and B . Solve them to find

$$M = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} \quad V = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Part 2. $\det M = -1/2$, so φ is odd and $1/2$ is the square of the ratio r , so $r = 1/2 \sqrt{2}$.

Part 3. Evidently,

$$X' = MX + B = 1/2 \sqrt{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \cos(-135^\circ) & \sin(-135^\circ) \\ -\sin(-135^\circ) & \cos(-135^\circ) \end{bmatrix} X + \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Therefore, φ is the composition of (first) the rotation about O with angle parameter -135° , then the reflection in the diagonal $x_1 = x_2$, the contraction with center O and ratio $1/2 \sqrt{2}$, and (last) the translation with vector $\langle 1, 1 \rangle$.

Part 4. To find the fixpoint X , solve equation $X = MX + B$ to get $X = (I - M)^{-1}B = \langle 6, -2 \rangle$. ♦

Exercise 27 asks you to classify four transformations. First, show that they're isometries. Then you have the same tools at hand as for earlier classification exercises. Use them! The logic is the same, but now you compute with coordinates.

Exercise 27. Show that each of the following systems of equations represents an isometry φ , and classify it.

$$\begin{array}{ll} \text{(a)} \quad \begin{cases} x'_1 = \frac{3}{5}x_1 + \frac{4}{5}x_2 \\ x'_2 = -\frac{4}{5}x_1 + \frac{3}{5}x_2 \end{cases} & \text{(c)} \quad \begin{cases} x'_1 = \frac{3}{5}x_1 + \frac{4}{5}x_2 + 1 \\ x'_2 = -\frac{4}{5}x_1 + \frac{3}{5}x_2 - 1 \end{cases} \\ \text{(b)} \quad \begin{cases} x'_1 = \frac{3}{5}x_1 - \frac{4}{5}x_2 \\ x'_2 = -\frac{4}{5}x_1 - \frac{3}{5}x_2 \end{cases} & \text{(d)} \quad \begin{cases} x'_1 = \frac{3}{5}x_1 - \frac{4}{5}x_2 + 1 \\ x'_2 = -\frac{4}{5}x_1 - \frac{3}{5}x_2 - 1 \end{cases} \end{array}$$

Solution. Call these transformations α to δ , respectively. Let τ be the translation with vector $\langle 1, -1 \rangle$. Let

$$A = \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ -\frac{4}{5} & \frac{3}{5} \end{bmatrix} \quad B = \begin{bmatrix} \frac{3}{5} & -\frac{4}{5} \\ -\frac{4}{5} & -\frac{3}{5} \end{bmatrix}.$$

You can check that $AA^t = BB^t = I$: these are orthogonal matrices. Thus α and β are isometries, and so are $\gamma = \tau\alpha$ and $\delta = \tau\beta$. Compute $\det A = 1$ and $\det B = -1$ to show that α and γ are even and β and δ are odd.

Since α fixes the origin O , it's a rotation about O with angle parameter θ . Thus A is the matrix A_θ defined in section 6.8, so $\cos \theta = \frac{3}{5}$ and $\sin \theta = -\frac{4}{5}$, hence $\theta \approx -53.13^\circ$. By theorem 6.9.3, γ is also a rotation with angle parameter θ . Compute the coordinates x_1 and x_2 of its center X by solving the equation $X = \gamma(X)$:

$$\begin{cases} x_1 = \frac{3}{5}x_1 + \frac{4}{5}x_2 + 1 \\ x_2 = -\frac{4}{5}x_1 + \frac{3}{5}x_2 - 1, \end{cases} \quad \text{that is,} \quad \begin{cases} -\frac{2}{5}x_1 + \frac{4}{5}x_2 = -1 \\ -\frac{4}{5}x_1 - \frac{2}{5}x_2 = 1. \end{cases}$$

You get $X = \langle -\frac{1}{2}, -\frac{3}{2} \rangle$.

Since β fixes the origin, it's a reflection across a line g through O . Points X on g are fixed:

$$\begin{cases} x_1 = \frac{3}{5}x_1 - \frac{4}{5}x_2 \\ x_2 = -\frac{4}{5}x_1 - \frac{3}{5}x_2, \end{cases} \quad \text{that is,} \quad \begin{cases} -\frac{2}{5}x_1 - \frac{4}{5}x_2 = 0 \\ -\frac{4}{5}x_1 - \frac{8}{5}x_2 = 0. \end{cases}$$

This system of equations is redundant. Its solution consists of all points $X = \langle x_1, x_2 \rangle$ such that $x_1 = -2x_2$. Thus, g is the line with equation $X = tP$, where $P = \langle -2, 1 \rangle$.

The glide reflection δ maps the origin to $O' = \langle 1, -1 \rangle$ and the point $U = \langle 1, 0 \rangle$ to $U' = \langle \frac{8}{5}, -\frac{9}{5} \rangle$. By exercise 11, midpoints $Q = \frac{1}{2}(O + O') = \langle \frac{1}{2}, -\frac{1}{2} \rangle$ and $R = \frac{1}{2}(U + U') = \langle \frac{13}{10}, -\frac{9}{10} \rangle$ lie on its axis h , which therefore has equation $X = Qt + R(1 - u)$. You can simplify that to $X = Pt + R$. Finally, δ has vector $Q' - Q = -\frac{3}{5}P$. ♦

The next three exercises could have been placed under the earlier **Classifying isometries** heading. To solve them you don't really need to work with their equations. But that might be the simplest way.

Exercise 28. Classify the compositions of the 1° rotation about point $\langle 2, 3 \rangle$ and the translation with vector $\langle 4, 5 \rangle$, in both orders.

Solution. Let $\theta = 1^\circ$ and ρ_θ be the rotation through angle θ about the origin. Let τ be the translation with vector $P = \langle 2, 3 \rangle$. By theorem 6.9.11, $\varphi = \tau\rho_\theta\tau^{-1}: X \rightarrow A_\theta(X - P) + P$ is the rotation through angle θ about P . Let ν be the translation with vector $V = \langle 4, 5 \rangle$. Then

$$\varphi\nu: X \rightarrow A_\theta(X + V - P) + P \quad \nu\varphi: X \rightarrow A_\theta(X - P) + P + V.$$

By theorem 6.9.3, they're both rotations through angle θ . Their centers Y and Z are their fixpoints, which you can compute by solving linear systems:

$$\begin{aligned} Y &= A_\theta(Y + V - P) + P & Z &= A_\theta(Z - P) + P + V \\ (A_\theta - I)Y &= A_\theta(P - V) - P(A_\theta - I) & Z &= A_\theta P - P - V \\ Y &\approx \langle -286.47, 229.68 \rangle & Z &= \langle -282.47, 234.68 \rangle. \blacklozenge \end{aligned}$$

Exercise 29. Classify the compositions of the 1° rotation about point $\langle 2, 3 \rangle$ and the 4° rotation about point $\langle 5, 6 \rangle$, in both orders.

Solution. Let $\eta = 4^\circ$ and ν be the translation with vector $Q = \langle 5, 6 \rangle$. By theorem 6.9.11, $\chi = \nu\rho_\eta\nu^{-1}: X \rightarrow A_\eta(X - Q) + Q$ is the rotation through angle η about Q . Consider the rotation φ in the exercise 28 solution. By theorem 6.9.3, the compositions

$$\begin{aligned} \psi &= \varphi\chi: X \rightarrow A_\theta A_\eta(X - Q) + A_\theta(Q - P) + P \\ \omega &= \chi\varphi: A_\eta A_\theta(X - P) + A_\eta(P - Q) + Q \end{aligned}$$

are both 5° rotations. Their centers Y and Z are their fixpoints, which you can compute by solving linear systems:

$$\begin{aligned} Y &= A_\theta A_\eta(Y - Q) + A_\theta(Q - P) + P \\ (A_\theta A_\eta - I)Y &= A_\theta A_\eta Q + A_\theta(P - Q) - P \\ Y &= \langle 4.3792, 5.4211 \rangle \end{aligned}$$

$$\begin{aligned} Z &= A_\eta A_\theta(Z - P) + A_\eta(P - Q) + Q \\ (A_\eta A_\theta - I)Z &= A_\eta A_\theta P + A_\eta(Q - P) - Q \\ Z &= \langle 4.4211, 5.3792 \rangle. \blacklozenge \end{aligned}$$

Exercise 30. Classify the compositions of the 10° rotation about the origin and the reflection across the line with equation $x_1 + x_2 = 1$, in both orders.

Solution. To find the equation of the reflection, apply exercise 24, part (2), with $P = \langle 1, 1 \rangle$ and $Q = \langle 1, 0 \rangle$, getting $P'P = 2$ and

$$K = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad KQ = P \quad A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

Reflection σ maps $X \rightarrow AX + P$ and rotation ρ maps $X \rightarrow A_{10^\circ}X$. Thus $\sigma\rho$ has matrix representation $X \rightarrow X' = AA_{10^\circ}X + P$. It's a glide reflection. The midpoints $R = \frac{1}{2}(P + P')$ and $S = \frac{1}{2}(Q + Q')$ lie on its axis. Numer-

ical software provides the axis equation $x_2 = -(\tan 50^\circ)x_1 + 1.0958$. The vector is $R' - R = \langle -0.07923, 0.09442 \rangle$.

$\rho\sigma$ has representation $X \rightarrow X' = A_{10^\circ}AX + A_{10^\circ}P$. It's also a glide reflection. The midpoints $R = \frac{1}{2}(P + P')$ and $S = \frac{1}{2}(Q + Q')$ lie on its axis. Numerical software provides the axis equation $x_2 = -(\tan 40^\circ)x_1 + 0.9195$. The vector is $R' - R = \langle -0.09442, 0.07923 \rangle$. ♦

A geometric transformation φ associates with points of a figure those of a corresponding figure—its image—perhaps in another place. You could call this the *alibi* version of the concept—*alibi* is the Latin word for *elsewhere*. There's another version, often used in analytic geometry. Suppose the figure is a coordinate system; then so is its image. A point P with coordinates $\langle x_1, x_2 \rangle$ relative to the first system has coordinates $\langle x'_1, x'_2 \rangle$ relative to the second. Associated with φ is a *coordinate transformation* that maps the coordinate pair $\langle x_1, x_2 \rangle$ to $\langle x'_1, x'_2 \rangle$. Some texts regard *that* as the basic geometric transformation concept: Points of a figure Φ have two corresponding coordinate pairs. You could call it the *alias* version—*alias* is the Latin word for *otherwise*, to which we often add *known as*. Suppose you want to use two coordinate systems to model different aspects of some phenomenon. It's usually easy to find the isometry φ that transforms one into the other; it's part of the problem framework. Then you can derive an equation for φ . How do you use that to find equations that relate the corresponding coordinate pairs $\langle x_1, x_2 \rangle$ and $\langle x'_1, x'_2 \rangle$ of a point P ? (These considerations were introduced, with slightly different notation, under the heading **Transforming coordinate systems** in section 6.9.) Figures Φ in your model may be described by equations that relate the two coordinates of points P of Φ . How do you convert an equation that relates x_1 and x_2 to one for x'_1 and x'_2 ? Exercise 31 provides some examples of this process. You'll have to decide when and how to use the equation of φ and that of its inverse. You'll see that how you transform equations that describe figures depends on whether they're single equations or pairs of parametric equations.

Exercise 31. Suppose that, in addition to your standard Cartesian coordinate system with origin O , you're using an auxiliary one with

- origin $O' = \langle 3, 2 \rangle$,
- unit point U' for the first axis located below and left of O' on the line through O' with slope 1, and
- unit point V' for the second axis located above and left of O' .

A point $X = \langle x_1, x_2 \rangle$ has coordinates x'_1 and x'_2 relative to this coordinate system.

Part 1. Find formulas for x'_1 and x'_2 in terms of x_1 and x_2 .

Part 2. Find the equation of the unit circle Γ with center O in terms of the auxiliary coordinate system: X must lie on Γ if and only if x'_1 and x'_2 satisfy the equation.

Part 3. $x_2 = x_1^2$ is the equation of a parabola Π . Find the equation of Π in terms of the auxiliary coordinate system.

Part 4. $x_1 = \cos t$ and $x_2 = 2 \sin t$ for $0 \leq t < 2\pi$ are parametric equations for an ellipse $\mathcal{A}$. Find parametric equations for $\mathcal{A}$ in terms of the auxiliary coordinate system.

Solution, part 1. The isometry $X \rightarrow X'$ that maps O' , U' , and V' to O , $U = \langle 1, 0 \rangle$, and $V = \langle 0, 1 \rangle$ is $\sigma_g \rho \tau_{-O'}$, where g is the second coordinate axis and ρ is the rotation with angle parameter -45° about O . That is,

$$X' = \begin{bmatrix} x'_1 \\ x'_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(-45^\circ) & -\sin(-45^\circ) \\ \sin(-45^\circ) & \cos(-45^\circ) \end{bmatrix} \begin{bmatrix} x_1 - 3 \\ x_2 - 2 \end{bmatrix}$$

$$\begin{cases} x'_1 = -\frac{1}{2}x_1\sqrt{2} - \frac{1}{2}x_2\sqrt{2} + \frac{5}{2}\sqrt{2} \\ x'_2 = -\frac{1}{2}x_1\sqrt{2} + \frac{1}{2}x_2\sqrt{2} + \frac{1}{2}\sqrt{2} \end{cases}$$

$$\begin{cases} x_1 = -\frac{1}{2}x'_1\sqrt{2} - \frac{1}{2}x'_2\sqrt{2} + 3 \\ x_2 = -\frac{1}{2}x'_1\sqrt{2} + \frac{1}{2}x'_2\sqrt{2} + 2 \end{cases}.$$

Part 2. To form the auxiliary equation for Γ , substitute these expressions for x_1 and x_2 in its equation $x_1^2 + x_2^2 = 1$: you get $x_1'^2 + x_2'^2 - 5x'_1\sqrt{2} - x'_2\sqrt{2} + 12 = 0$.

Part 3. The same substitution in $x_2 = x_1^2$ yields the parabola's auxiliary equation: $x_1'^2 + 2x'_1x'_2 + x_2'^2 - 5x'_1\sqrt{2} - 7x'_2\sqrt{2} + 14 = 0$.

Part 4. To derive the auxiliary equations for the ellipse, substitute the given parametric equations into the equations for x'_1 and x'_2 : you get

$$\begin{cases} x'_1 = -\frac{1}{2}\sqrt{2}\cos t - \sqrt{2}\sin t + \frac{1}{2}\sqrt{2} \\ x'_2 = -\frac{1}{2}\sqrt{2}\cos t + \sqrt{2}\sin t + \frac{1}{2}\sqrt{2} \end{cases} \cdot \blacklozenge$$

Reflection calculus

During the 1920s Gerhard Thomsen discovered how to formulate many familiar properties of plane figures in terms of equations involving compositions of point and line reflections. The first few exercises under this heading introduce this method. You can solve them by manipulating equations

involving reflections, half turns, translations, and occasionally vector algebra. You never have to use coordinates explicitly. You can call this method a *reflection calculus*.

Exercise 32. Show that for any points O , P , and Q , the composition $\sigma_O \sigma_P \sigma_Q$ is a half turn about point $R = O - P + Q$, and $OPRQ$ is a (possibly degenerate) parallelogram.

Solution. Since $O - P = R - Q$, $OPRQ$ is a (possibly degenerate) parallelogram. Translations $\sigma_O \sigma_P$ and $\sigma_R \sigma_Q$ are equal by corollary 6.4.5, hence $\sigma_O \sigma_P \sigma_Q = \sigma_R$. ♦

Exercise 33. Describe the figure formed by points A , B , C , and G for which $\sigma_A \sigma_G \sigma_B \sigma_G \sigma_C \sigma_G = \iota$. *Suggestion:* Reason with vectors.

Solution. This equation is equivalent to $\tau_{2(A-G)} \tau_{2(B-G)} \tau_{2(C-G)} = \iota$, to $2(A - G) + 2(B - G) + 2(C - G) = O$, to $A + B + C = 3G$, and to $G = \frac{1}{3}(A + B + C)$. Hence the figure consists of $\triangle ABC$ and its centroid G when A , B , and C are noncollinear; otherwise it consists of three collinear points and their “average”. ♦

Exercise 34. Construct a pentagon with given edge midpoints M_1 to M_5 . Is it unique? Generalize your solution to apply to polygons with more edges.

Solution. Denote the pentagon’s vertices by P_0 to P_4 , and set $P_5 = P_0$, so that each edge $\overline{P_{i-1}P_i}$ has midpoint M_i . Then P_0 is a fixpoint of

$$\sigma_{M_1} \sigma_{M_2} \sigma_{M_3} \sigma_{M_4} \sigma_{M_5} = \sigma_{M_1 - M_2 + M_3} \sigma_{M_4} \sigma_{M_5} = \sigma_{M_1 - M_2 + M_3 - M_4 + M_5},$$

Gerhard THOMSEN was born in Hamburg in 1899, the son of a physician. He served a year in World War I, then became one of the first students of the new university at Hamburg. Thomsen completed the Ph.D. in 1923 with a dissertation on differential geometry. He served as assistant in Karlsruhe and Hamburg, studied a year with Tullio Levi-Civita in Rome, then presented his Habilitationsschrift in Hamburg, on a problem in gravitational physics. In 1929 he became Ausserordentlicher Professor at Rostock. There he continued work on differential geometry, collaborating with Wilhelm Blaschke, and published as well in mathematical physics and foundations of geometry. In 1933, reacting to the Nazification turmoil in his university, he published an inflammatory lecture that seemed to support some Nazi aims but attacked Nazi suppression of education in the sciences. This evidently attracted the attention of the secret police. Thomsen died, an apparent suicide, on a Rostock railroad track in 1934.

hence $P_0 = M_1 - M_2 + M_3 - M_4 + M_5$. Further, $P_{i+1} = \sigma_{M_{i+1}}(P_i)$ for $i = 0$ to 3 . The pentagon is unique. This technique generalizes to determine a polygon with any odd number of sides, given any prescribed set of edge midpoints. ♦

Exercise 35. When can you construct a quadrilateral with given edge midpoints? When is it unique? How about a hexagon? *Suggestion:* What kind of figure is formed by the midpoints of the edges of a quadrilateral? Of a hexagon?

Solution. By exercise 4.5.2 the edge midpoints of a quadrilateral form a parallelogram. Conversely, if points M_1 to M_4 form a parallelogram, then $M_1 - M_2 = M_4 - M_3$, so $M_1 - M_2 + M_3 - M_4 = O$, hence any point P_0 is a fixpoint of

$$\sigma_{M_1} \sigma_{M_2} \sigma_{M_3} \sigma_{M_4} = \tau_{2(M_1 - M_2)} \tau_{2(M_3 - M_4)} = \tau_{2(M_1 - M_2 + M_3 - M_4)} = \tau_0 = \iota.$$

You can build a quadrilateral $P_0 \dots P_3$ whose edge midpoints are M_1 to M_4 by setting $P_{i+1} = \sigma_{M_{i+1}}(P_i)$ for $i = 0$ to 2 .

The edge midpoints M_1 to M_6 of a hexagon $P_0 \dots P_5$ satisfy equation $M_1 - M_2 + M_3 - M_4 + M_5 - M_6 = O$, because P_0 is a fixpoint of the translation

$$\sigma_{M_1} \sigma_{M_2} \sigma_{M_3} \sigma_{M_4} \sigma_{M_5} \sigma_{M_6} = \tau_{2(M_1 - M_2 + M_3 - M_4 + M_5 - M_6)},$$

which must therefore be the identity. Conversely, if points M_1 to M_6 satisfy that equation, then you can build a hexagon $P_0 \dots P_5$ whose edge midpoints are M_1 to M_6 by choosing any point P_0 and setting $P_{i+1} = \sigma_{M_{i+1}}(P_i)$ for $i = 0$ to 4 . ♦

Exercises 34 and 35 are closely related to exercise 17 on billiards and to exercise 18. Can you imagine the pentagon and hexagon as *dual billiards* tables? You play on the outside, grazing the corners instead of rebounding from the sides!

A development of the reflection calculus and results obtained with it were published in Thomsen 1933a. Exercises 36 to 39 are selected from section 4 of that book.

Exercise 36. Consider points P and Q , lines g and h , and the isometry $\varphi = \sigma_g \sigma_P \sigma_h \sigma_Q$. Under what condition on the points and lines is $\varphi = \iota$? *Suggestion:* First, classify $\sigma_g \sigma_P$ and $\sigma_h \sigma_Q$.

Solution. Find the feet O and R of the perpendiculars k and l to g and h through P . Then

$$\begin{aligned} \sigma_g \sigma_P &= \sigma_g \sigma_O \sigma_O \sigma_P = \sigma_k \tau_{2(O-P)} \\ \sigma_h \sigma_Q &= \sigma_h \sigma_R \sigma_R \sigma_Q = \sigma_l \tau_{2(R-Q)}. \end{aligned}$$

These are glide reflections with axes k and l and vectors $2(O - P)$ and $2(R - Q)$. Now, $\varphi = \iota$ if and only if $\sigma_g \sigma_P = (\sigma_h \sigma_Q)^{-1}$. This condition is equivalent to $k = l$ and $2(O - P) = -2(R - Q)$, hence it holds if and only if O, P, Q , and R are collinear and $O - P = Q - R$. ♦

Exercise 37. Prove that these conditions on lines g and h are equivalent:

- (a) $g \parallel h$;
- (b) for every point P , $\sigma_g \sigma_h \sigma_P \sigma_g \sigma_h \sigma_P = \iota$ but $\sigma_g \sigma_h \sigma_P \neq \iota$;
- (c) for some point P , $\sigma_g \sigma_h \sigma_P \sigma_g \sigma_h \sigma_P = \iota$ but $\sigma_g \sigma_h \sigma_P \neq \iota$.

Solution. Assume $g \parallel h$ and consider an arbitrary point P . Let Q and R be the intersections of g and h with the perpendicular k to g and h through P . By corollary 6.4.5, $\sigma_g \sigma_h = \sigma_Q \sigma_R$. By exercise 32, $\sigma_g \sigma_h \sigma_P = \sigma_Q \sigma_R \sigma_P = \sigma_O$ for some point O . Since $\sigma_O \sigma_O = \iota$ but $\sigma_O \neq \iota$, Condition (a) implies condition (b). Clearly, condition (b) implies condition (c).

To prove that condition (c) implies condition (a), assume that (c) holds and (a) fails, and deduce a contradiction as follows. Under that assumption, $\rho = \sigma_g \sigma_h$ would be a nonidentity rotation with angle parameter θ , and by theorem 6.9.3, $\sigma_g \sigma_h \sigma_P = \rho \sigma_P$ would be a translation τ_V with nonzero vector V , or else a rotation with angle parameter $\theta + 180^\circ$. Accordingly, $\rho \sigma_P \rho \sigma_P$ would be τ_{2V} or else a rotation with angle parameter 2θ . Is either alternative consistent with the assumption $\rho \sigma_P \rho \sigma_P = \iota$ but $\rho \sigma_P \neq \iota$? The translational case leads to $\tau_{2V} = \iota$ but $\tau_V \neq \iota$, which is clearly impossible. The rotational case leads to $2\theta \equiv 0^\circ$ but $\theta \not\equiv 0^\circ \pmod{360^\circ}$, hence $\theta \equiv 180^\circ$ and $\rho \sigma_P = \sigma_O$ for some point O ; but then ρ would have to be the translation $\sigma_O \sigma_P$, which is impossible. The assumptions have indeed led to a contradiction. ♦

Exercise 38. Consider some lines a , b , and c and the isometry

$$\varphi = \sigma_c \sigma_a \sigma_b \sigma_a \sigma_b \sigma_a \sigma_c \sigma_b \sigma_a \sigma_c \sigma_a \sigma_c \sigma_b.$$

Prove that φ is the identity if and only if a , b , and c are copencilar or form an isosceles triangle with congruent edges on b and c . *Suggestion:* It's easy to show that copencilarity implies $\varphi = \iota$. When there's a triangle, consider the line g through the feet B^* and C^* of the altitudes perpendicular to b and c . When is $g \parallel a$? What isometry is $\gamma = \sigma_b \sigma_a \sigma_c$?

Solution. If a , b , and c are copencilar, then $\gamma = \sigma_d$ for some line d , by corollary 6.4.7 or 6.5.6. Therefore,

$$\varphi = \sigma_d^{-1} \sigma_a \sigma_d \sigma_d \sigma_a \sigma_d^{-1} = \sigma_d^{-1} \sigma_a \sigma_a \sigma_d^{-1} = \sigma_d^{-1} \sigma_d^{-1} = \sigma_d \sigma_d = \iota.$$

Now assume they form a triangle. By theorem 6.7.2, γ is a glide reflection with axis g and vector V parallel to g : $\gamma = \tau_V \sigma_g$ and $\gamma \gamma = \tau_{2V}$. Thus, $\varphi = \gamma^{-1} \sigma_a \gamma \gamma \sigma_a \gamma^{-1}$, and $\varphi = \iota$ if and only if

and $\mathcal{A}$ have two distinct intersections X_1 and X_2 . Then the circles Γ' and $\mathcal{A}'$ with centers P' and Q' and radii r and s have two distinct intersections Y and Z , because $r + s > P'Q' = PQ$. But the two distinct images X'_1 and X'_2 also lie on $\Gamma' \cap \mathcal{A}'$, because $P'X'_i = PX_i = r$ and $Q'X'_i = QX_i = s$ for $i = 1$ and 2 . Therefore, either $X'_1 = Y$ and $X'_2 = Z$ or $X'_1 = Z$ and $X'_2 = Y$. ♦

Exercise 41. Let r be any positive number and φ be a function from the plane to itself such that $X'Y' = r(XY)$ for all X and Y . Prove that φ is a similarity with ratio r . *Suggestion:* Use exercise 40.

Solution. Let ψ be any similarity with ratio r^{-1} . Then $\chi = \psi\varphi$ is a function from the plane to itself that preserves distances. By exercise 40, χ is an isometry, so $\varphi = \psi^{-1}$ is a similarity with ratio r . ♦

It's possible to weaken the definitions of isometry and similarity even further than these exercises suggest. For example, invariance of the property of separation by distance 1 —that is, the condition $XY = 1$ if and only if $X'Y' = 1$ —implies that φ is an isometry.⁶

Dilatations

A *dilatation*⁷ is a plane transformation $\varphi : X \rightarrow X'$ such that the image $g' = \varphi[g]$ of any line g is a line parallel or equal to g . The last four exercises of this chapter develop the theory of dilatations, which plays a major role in some axiomatizations of geometry.⁸ You should solve them without using any limit concepts from analysis. Your solutions will provide a proof of the fixpoint theorem for similarities, theorem 6.10.8, that, unlike the proof in the text, requires no analysis. In your solutions of exercises 43 to 45, you're expected to use the results in the preceding exercises.

Exercise 42. Prove that the dilatations form a transformation group. Of the various types of isometries and similarities considered in this chapter, which are dilatations and which not?

Solution. That the dilatations form a group follows from the fact that *parallel or equal* is an equivalence relation on the set of all lines. The dilatation group includes all translations, half turns, contractions and dilations, but no glide reflections, and no rotations except half turns. ♦

⁶ Modenov and Parkhomenko [1961] 1965, appendix.

⁷ Be careful not to confuse the terms *dilation* and *dilatation*! The first stems from the Latin prefix *dis-*, meaning *apart*, with the past participle *latus* of the irregular Latin verb *fero*, which means *carry*. The second is from the related regular verb *dilato*, which means *broaden*.

⁸ For example, Artin 1957, chapter 2.

Exercise 43. Show that

- (a) the only dilatation with two distinct fixpoints is the identity, and
- (b) if φ and χ are dilatations and $\varphi(X) = \chi(X)$ for two distinct points X , then $\varphi = \chi$.

Suggestions: For (a) show first that a dilatation $\varphi : X \rightarrow X'$ with distinct fixpoints P and Q fixes every point *not* on $\overleftrightarrow{PQ}$, then consider points *on* the line. Note that any line through a fixpoint is fixed and any point on two fixed lines is fixed.

Solution. Suppose $P \neq Q$, $P' = P$, and $Q' = Q$. Let R be any point noncollinear with P and Q , and consider the lines $g = \overleftrightarrow{PR}$ and $h = \overleftrightarrow{QR}$. Then $g' = g$ because g' is parallel or equal to g and P lies on g and g' . Similarly, $h' = h$, so R and R' lie on g and h . If $R' \neq R$, then $g = h$, and R would be collinear with P and Q , contradiction! Thus φ fixes every point R noncollinear with P and Q . Now suppose S is a point collinear with P and Q . The argument of the previous paragraph with Q and R replaced by R and S implies that φ fixes S . Part (a) is proved.

To prove (b), note that $\varphi^{-1}\chi$ has two distinct fixpoints, so it's the identity. ♦

Exercise 44. Show that every dilatation $\varphi : X \rightarrow X'$ that's not a translation has a fixpoint. *Suggestion:* Find P and Q such that $PP'Q'Q$ is a quadrilateral, then show that it can't be a parallelogram.

Solution. Suppose φ is a dilatation but not a translation. There's a point P for which $P' \neq P$; define $g = \overleftrightarrow{PP'}$. Then P' lies on g and g' , so $g' = g$. Select any point Q not on g , so that $P \neq Q$, and define $h = \overleftrightarrow{PQ}$. Then $h' = \overleftrightarrow{P'Q'} \neq h$ and $Q' \neq Q$. Define $k = \overleftrightarrow{QQ'}$. Then Q' lies on k and k' , so $k = k'$. If k were parallel or equal to g , then $PP'Q'Q$ would be a (perhaps degenerate) parallelogram, there'd be a translation τ such that $\tau(P') = P$ and $\tau(Q') = Q$, and the dilatation $\tau\varphi$ would have distinct fixpoints P and Q ; by exercise 43, $\tau\varphi$ would be the identity and φ would be the translation τ^{-1} , contradiction! Therefore, k is neither parallel nor equal to g , so $g \cap k$ is a point O . Then $O = O'$ because O and O' both lie on $g = g'$ and on $k = k'$. ♦

Exercise 45. Prove that every dilatation that's not a translation is a contraction, a dilation, or the composition of a contraction or dilation with the half turn about its center.

Solution. Suppose $\varphi : X \rightarrow X'$ is a dilatation but not a translation. By exercise 44 it has a fixpoint O . Select any point $P \neq O$, and consider the contraction or dilation ψ with center O and ratio $r = OP'/OP$, so that $P' = \psi(P)$ or $P' = \sigma_O\psi(P)$. Then the dilatation $\varphi^{-1}\psi$ or $\varphi^{-1}\sigma_O\psi$ has distinct fixpoints O and P , so it's the identity. Thus $\varphi = \psi$ or $\sigma_O\psi$. ♦

7.7

Solutions to exercises on three-dimensional isometries and similarities

Concepts

- Distance-preserving functions
- Dilatations
- Compositions of commuting point and plane reflections and half turns
- Conjugates
- Compositions of reflections, translations, and rotations
- Classifying isometries and similarities, given their equations
- Finding fixpoints and fixed lines of isometries, given their equations
- Reflection calculus
- Three-reflections theorems

This section contains thirty-two exercises related to the material covered earlier in this chapter. The first three ask for some straightforward proofs omitted earlier. Several others, listed under the headings **Commuting reflections and half turns**, **Conjugacy exercises**, and **Classifying isometries**, extend

the chapter's methods beyond the essentials covered earlier, and provide a nearly complete treatment of the interrelationships among reflections, translations, and rotations. The conjugacy exercises firmly establish results you'll often find used without mention in applications. The exercises under the heading **Exercises on equations** familiarize you with the algebraic and numerical computations often required to apply the theory. Some require tedious arithmetic to evaluate determinants and solve linear systems. There was no effort to avoid that. You're expected to use mathematical software for that work. Finally, exercises 5 and 6 and those under the heading **Reflection calculus** provide excursions into areas less central to the program of this book. You've already met their two-dimensional counterparts in chapter 6.

General exercises

Exercises 1 to 3 ask for several proofs omitted from earlier sections of this chapter.

Exercise 1. Prove theorem 7.1.4: The only isometry with four noncoplanar fixpoints is the identity. *Suggestion:* Imitate the proof of theorem 6.2.7.

Solution. Find a plane ε through three noncollinear fixpoints, a fixpoint P not on ε , and the plane δ through P parallel to ε . By theorem 7.1.3, each point on ε is fixed. Each point X not on δ or ε lies on line $P\vec{Q}$ for some point Q in ε . By theorem 7.1.2, X is fixed. Finally, each point Y on δ lies on line $Q\vec{X}$ for some points Q in ε and X not on δ or ε . By theorem 7.1.2, Y is fixed. ♦

Exercise 2. Prove theorem 7.4.3: A rotation ρ commutes with a reflection σ_ε just when $\rho = \iota$, or its axis g is perpendicular to the mirror ε , or it's a half turn about a line in ε . *Suggestion:* Imitate the proof of theorem 7.4.1.

Solution. Showing that the rotation and reflection commute when any of these conditions holds is straightforward. Suppose they all fail. First, suppose $\rho \neq \iota$, g is not perpendicular to ε , and $g \not\subseteq \varepsilon$. Then $\rho\sigma_\varepsilon(P) \neq \sigma_\varepsilon\rho(P)$ for any point P on g but not on ε . Second, suppose $\rho \neq \iota$ and g lies in ε but ρ isn't self-inverse. Then $\rho\sigma_\varepsilon(Q) \neq \sigma_\varepsilon\rho(Q)$ for any point Q in ε but not on g . ♦

Exercise 3. Prove theorem 7.5.1: A translation τ_V commutes with a rotation ρ just when $V = O$, $\rho = \iota$, or $\vec{OV}$ is parallel to the axis of ρ . *Suggestion:* Imitate the proofs of theorems 7.4.2 and 7.4.4.

Solution. Showing that τ_V and ρ commute when any of these conditions holds is straightforward. If they all fail, then $\tau_V\rho$ and $\rho\tau_V$ have different effects on any point on the axis. ♦

Exercise 4 is the three-dimensional analog of exercise 6.11.8.

Exercise 4. Show that an isometry φ is even if and only if $\varphi = \chi\chi$ for some isometry χ .

Solution. An even isometry φ is a screw with vector V and angle parameter θ . Let χ be the screw with vector $\frac{1}{2}V$ and angle parameter $\frac{1}{2}\theta$. Then $\varphi = \chi\chi$. Conversely, $\chi\chi$ is even no matter what the parity of χ . ♦

Exercises 5 and 6 pursue excursions into less central themes, already begun in two-dimensional exercises under the section 6.11 headings **Exercises on the definitions of isometry and similarity** and **Dilatations**. If you've solved those in the most efficient way, you should need little to adapt your work to three dimensions.

Exercise 5. Formulate and prove three-dimensional versions of exercises 6.11.40–6.11.41 on the definitions of isometry and similarity.

Solution. The analog of exercise 6.11.40 is true. $X' = Y'$ implies $0 = X'Y' = XY$, hence $X = Y$; therefore φ is injective. To prove that it's surjective, select any point Y . You must find a point X such that $X' = Y$. First, select any noncollinear points P , Q , and R . Their images P' , Q' , and R' aren't collinear; if they were, then one—for example Q' —would lie between the other two, which would imply $PQ + QR = P'Q' + Q'R' = P'R' = PR$, which can't be true. Consider the spheres with centers P , Q , and R and radii $r = P'Y$, $s = Q'Y$, and $t = R'Y$. Their intersection is a finite point set Σ . The intersection Σ' of the corresponding spheres with centers P' , Q' and R' and the same radii contains the same number of points. (Verify that in detail!) Since φ maps Σ into Σ' injectively, φ must map Σ onto Σ' . Since $Y \in \Sigma'$, there's a point $X \in \Sigma$ such that $Y = \varphi(X)$.

The analog of Exercise 6.11.41 is also true; its solution is valid in three dimensions. ♦

Exercise 6. Formulate and prove three-dimensional analogs of exercises 6.11.42–6.11.45 on dilatations.

Solution. See exercises 6.11.42 to 45 and their solutions. The entire theory of dilatations is independent of dimension. ♦

Commuting reflections and half turns

The conditions in plane geometry under which reflections and half turns commute—the nontrivial self-inverse isometries—were laid out at the end of section 6.3. Those ideas played essential roles in the rest of that chapter. In three-dimensional geometry there are *three* types of nontrivial self-inverse isometries. A complete treatment of their commutativity would have clogged the earlier sections of this chapter unacceptably, so they covered only a few aspects. In fact, the theory in this chapter was streamlined to allow that omission. But its *applications* require *all* the commutativity details, so they're presented here systematically, as exercises.

Exercise 7. Show that the composition $\sigma_P \sigma_Q$ of two point reflections is the translation with vector $2(P - Q)$, and they don't commute unless $P = Q$.

Solution. By theorem 7.4.4, $P = \frac{1}{2}(X + \sigma_P(X))$, so $\sigma_P(X) = 2P - X$, for all X . Similarly, $\sigma_Q(X) = 2Q - X$, so

$$\sigma_P \sigma_Q(X) = 2P - \sigma_Q(X) = 2P - (2Q - X) = X + 2(P - Q).$$

The reflections commute just when $2(P - Q) = 2(Q - P)$ —that is, $P = Q$. ♦

Exercise 8. Show that point and plane reflections σ_P and σ_ε commute just when P lies on ε , and in that case $\sigma_P \sigma_\varepsilon$ is the half turn about the line $g \perp \varepsilon$ through P .

Solution. Let $Q = g \cap \varepsilon$, so that $\sigma_P \sigma_\varepsilon(Q) = \sigma_P(Q)$. If P does *not* lie on ε , then neither does $\sigma_P(Q)$, by theorem 7.4.4; therefore $\sigma_\varepsilon \sigma_P(Q) \neq \sigma_P(Q)$, hence $\sigma_P \sigma_\varepsilon \neq \sigma_\varepsilon \sigma_P$. If P *does* lie on ε , then $\sigma_P = \sigma_g \sigma_\varepsilon = \sigma_\varepsilon \sigma_g$ by theorem 7.4.3, hence $\sigma_P \sigma_\varepsilon = \sigma_g = \sigma_\varepsilon \sigma_P$. ♦

Exercise 9. Show that

- Part 1.** a half turn σ_g commutes with a plane reflection σ_ε if and only if g lies in ε or $g \perp \varepsilon$;
- Part 2.** if g lies in ε , then $\sigma_g \sigma_\varepsilon$ is the reflection across the plane perpendicular to ε through g ;
- Part 3.** if $g \perp \varepsilon$, then $\sigma_g \sigma_\varepsilon$ is the reflection across $g \cap \varepsilon$.

Solution. The only part of this result that's not in theorem 7.4.3 is the classification of $\sigma_g \sigma_\varepsilon$ when g lies in ε . But that follows from $\sigma_\delta \sigma_\varepsilon = \sigma_g$. ♦

Exercise 10. Show that a point reflection σ_P and a half turn σ_g commute if and only if P lies on g , and in that case $\sigma_P \sigma_g$ is the reflection in the plane $\varepsilon \perp g$ through P .

Solution. Let $Q = g \cap \varepsilon$, so that $\sigma_P \sigma_g(Q) = \sigma_P(Q)$. If P does *not* lie on g , then neither does $\sigma_P(Q)$, by theorem 7.4.4; therefore $\sigma_g \sigma_P(Q) \neq \sigma_P(Q)$, hence $\sigma_P \sigma_g \neq \sigma_g \sigma_P$. If P *does* lie on g , then $\sigma_P = \sigma_g \sigma_\varepsilon = \sigma_\varepsilon \sigma_g$ by theorem 7.4.3, hence $\sigma_P \sigma_g = \sigma_\varepsilon = \sigma_P \sigma_g$. ♦

Exercise 11. Show that half turns σ_g and σ_h about distinct lines g and h commute if and only if $g \perp h$, and in that case, $\sigma_g \sigma_h$ is the reflection across the intersection of g and h .

Solution. Let k be the line perpendicular to g and h , and $P = h \cap k$, so that $\sigma_g \sigma_h(P) = \sigma_g(P)$. If g is not perpendicular to h , so that P does not lie on g , then neither does $\sigma_g(P)$, by theorem 7.4.4; therefore $\sigma_h \sigma_g(P) \neq \sigma_g(P)$, hence $\sigma_g \sigma_h \neq \sigma_h \sigma_g$. Now suppose $g \perp h$, so that P *does* lie on g , and consider the planes δ and ε through P perpendicular to g and h . Then $\sigma_g \sigma_h = \sigma_g \sigma_P \sigma_P \sigma_h = \sigma_\delta \sigma_\varepsilon = \sigma_k$ and $\sigma_h \sigma_g = \sigma_k^{-1} = \sigma_k = \sigma_g \sigma_h$. ♦

Conjugacy exercises

The conjugacy classes of two-dimensional isometries were described in complete detail under that heading in section 6.9. That information is important for applications, because authors often handle, for example, all 60° rotations interchangeably without mentioning explicitly the conjugacy relation that connects them. The same is true in three-dimensional applications. Earlier sections of chapter 7 contained the most essential conjugacy results: the general ideas near the end of section 7.1, and theorems 7.2.5, 7.3.3, and 7.3.5. Further results were omitted because their proofs are unwieldy and repetitive. They're offered here as exercises.

Exercise 12 shows that, like the plane reflections, the half turns and point reflections each constitute single conjugacy classes.

Exercise 12. Show that $\varphi \sigma_P \varphi^{-1} = \sigma_{\varphi(P)}$ and $\varphi \sigma_g \varphi^{-1} = \sigma_{\varphi[g]}$ for any point P , any line g , and any isometry φ . *Suggestion:* Consider fixpoints!

Solution. $\varphi \sigma_P \varphi^{-1}$ is odd and self-inverse. Adapt the proof of theorem 6.9.8 to show that $\varphi(P)$ is its sole fixpoint. Similarly, $\varphi \sigma_g \varphi^{-1}$ is even and self-inverse, and its fixpoints constitute the line $\varphi[g]$. And $\sigma_{\varphi(P)}$ and $\sigma_{\varphi[g]}$ are the only isometries with those properties. ♦

Exercise 13. Describe the conjugacy class of a glide reflection.

Solution. A glide reflection is an isometry $\gamma = \sigma_P \sigma_Q \sigma_\delta$, where points P and Q lie in plane δ . Its conjugate by an isometry $\varphi: X \rightarrow X'$ is

$$\varphi \gamma \varphi^{-1} = \varphi \sigma_P \sigma_Q \sigma_\delta \varphi^{-1} = \varphi \sigma_P \varphi^{-1} \varphi \sigma_Q \varphi^{-1} \varphi \sigma_\delta \varphi^{-1} = \sigma_{\varphi(P)} \sigma_{\varphi(Q)} \sigma_{\varphi[\delta]},$$

where P' and Q' lie in $\varphi[\delta]$. The conjugate is thus a glide reflection whose vector $2(P' - Q')$ has the same length as $2(P - Q)$, the vector of γ . Moreover, any glide reflection $\chi = \sigma_R \sigma_S \sigma_\varepsilon$ whose vector $2(R - S)$ has the same length is a conjugate. To prove that, first find a translation or rotation ψ such that $\psi[\delta] = \varepsilon$, then the translation τ such that $\tau\psi(P) = R$, and finally the rotation ρ about the line perpendicular to ε through R such that $\rho\tau\psi(Q) = S$. Now let $\varphi = \rho\tau\psi$, so that $\varphi(P) = R$, $\varphi(Q) = S$, and $\varphi[\delta] = \varepsilon$. It follows that $\chi = \varphi\gamma\varphi^{-1}$. In summary, *the conjugacy class of a glide reflection γ consists of all glide reflections with vectors whose length is the same as that of γ .* ♦

Exercise 14. Describe the conjugacy class of a rotary reflection.

Solution. A rotary reflection is an isometry $\omega = \sigma_\varepsilon \sigma_\zeta \sigma_\eta$, where ζ and η pass through a line $g \perp \varepsilon$. Its conjugate by an isometry φ is

$$\varphi\omega\varphi^{-1} = \varphi\sigma_\varepsilon\sigma_\zeta\sigma_\eta\varphi^{-1} = \varphi\sigma_\varepsilon\varphi^{-1}\varphi\sigma_\zeta\varphi^{-1}\varphi\sigma_\eta\varphi^{-1} = \sigma_{\varphi[\varepsilon]}\sigma_{\varphi[\zeta]}\sigma_{\varphi[\eta]},$$

where $\varphi[\zeta], \varphi[\eta] \perp \varphi[\varepsilon]$ and $\varphi[\zeta]$ and $\varphi[\eta]$ pass through $\varphi[g]$. The dihedral angles formed by $\varphi[\zeta]$ and $\varphi[\eta]$ have the same measures as those formed by ζ and η , so the conjugate is a rotary reflection with the same angle parameter as that of ω . Moreover, any rotary reflection $\chi = \sigma_\delta \sigma_\alpha \sigma_\beta$ with the same angle parameter is a conjugate. To prove that, first find a translation or rotation ψ such that $\psi[\varepsilon] = \delta$, then the translation τ such that $\tau\psi[g]$ is the intersection $h = \alpha \cap \beta$, and finally the rotation ρ about h such that $\rho\tau\psi[\zeta] = \alpha$ and $\rho\tau\psi[\eta] = \beta$. Now let $\varphi = \rho\tau\psi$, so that $\varphi[\delta] = \varepsilon$, $\varphi[\zeta] = \alpha$, and $\varphi[\eta] = \beta$. It follows that $\varphi\omega\varphi^{-1} = \chi$. In summary, *the conjugacy class of a rotary reflection consists of all rotary reflections with the same angle parameter.* ♦

Exercise 15. Describe the conjugacy class of a screw.

Solution. A screw is an isometry $\omega = \sigma_P \sigma_Q \sigma_\delta \sigma_\varepsilon$, where points P and Q lie in planes δ and ε . Its conjugate by an isometry $\varphi: X \rightarrow X'$ is

$$\begin{aligned}\varphi\omega\varphi^{-1} &= \varphi\sigma_P\sigma_Q\sigma_\delta\sigma_\varepsilon\varphi^{-1} = \varphi\sigma_P\varphi^{-1}\varphi\sigma_Q\varphi^{-1}\varphi\sigma_\delta\varphi^{-1}\varphi\sigma_\varepsilon\varphi^{-1} \\ &= \sigma_{\varphi[P]}\sigma_{\varphi[Q]}\sigma_{\varphi[\delta]}\sigma_{\varphi[\varepsilon]},\end{aligned}$$

where P' and Q' lie in $\varphi[\delta]$ and $\varphi[\varepsilon]$. The conjugate is thus a screw whose vector $2(P' - Q')$ has the same length as $2(P - Q)$, the vector of ω . The dihedral angles formed by $\varphi[\delta]$ and $\varphi[\varepsilon]$ have the same measures as those formed by δ and ε , so the screws have the same angle parameter. Moreover, any screw $v = \sigma_R \sigma_S \sigma_\zeta \sigma_\eta$ with the same angle parameter whose vector $2(R - S)$ has the same length is a conjugate. To prove that, first find a translation or rotation ψ such that $\psi[\delta] = \zeta$, then the translation τ such that $\tau\psi(P) = R$, then the rotation ρ about the perpendicular to ζ through R such that $\rho\tau\psi(Q) = S$. Finally, let χ be the identity or the

reflection across a plane through R and S so that $\chi\rho\tau\psi[\varepsilon] = \eta$, and set $\varphi = \chi\rho\tau\psi$, so that $\varphi(P) = R$, $\varphi(Q) = S$, and $\varphi[\delta] = \eta$. It follows that $v = \varphi\gamma\varphi^{-1}$. In summary, the conjugacy class of a screw ω consists of all screws with the same angle parameter, whose vectors have the same length as that of ω . ♦

Classifying isometries

Earlier in this chapter, a glide reflection was defined as a composition of a translation and a commuting plane reflection, a rotary reflection as a composition of a rotation and a commuting plane reflection, and a screw as a composition of a translation and a commuting rotation. With these notions, you can classify *all* isometries. Exercises 16 to 18 give you some practice. What kinds of isometries are the analogous compositions whose components *don't* commute?

Exercise 16. What's the composition of a translation and a plane reflection when they don't commute?

Solution. Find distinct points P and Q and a plane γ through Q such that the composition is $\varphi = \sigma_P\sigma_Q\sigma_\gamma$. Find the planes α and β through P and Q perpendicular to $\overleftrightarrow{PQ}$, so that $\varphi = \sigma_\alpha\sigma_\beta\sigma_\gamma$. Then α , β , and γ have a common perpendicular plane, so by theorem 7.4.2, φ is a glide reflection. (γ is *not* its mirror.) If α , β , and γ happened to be parallel, φ would be a plane reflection. ♦

Exercise 17. What's the composition of a rotation and a plane reflection when they don't commute?

Solution. Find a plane ε and distinct planes γ and δ through a line g such that the composition is $\varphi = (\sigma_\gamma\sigma_\delta)\sigma_\varepsilon$. If $g \parallel \varepsilon$ then γ , δ , and ε would have a common perpendicular plane, and by theorem 7.4.2 φ would be a glide reflection. By theorem 7.4.3, that happens when g doesn't lie in ε or when g lies in ε but γ and δ aren't perpendicular. In that second case, φ would be a reflection. By theorem 7.4.3 it's possible that $g \cap \varepsilon$ is a point, so that γ , δ , and ε have no common perpendicular plane; in that case theorem 7.4.5 would imply that φ is a rotary reflection. ♦

Exercise 18. What's the composition of a translation and a rotation when they don't commute?

Solution. By theorem 7.5.3, the composition is a screw. It could be a rotation—select distinct planes γ , δ , and ε with a common perpendicular plane such that γ and δ are parallel but δ and ε are not. Then $\sigma_\gamma\sigma_\delta$ is a translation and $\sigma_\delta\sigma_\varepsilon$ is a rotation, but they don't commute and their

composition is the rotation $\sigma_\gamma \sigma_\delta \sigma_\delta \sigma_\epsilon = \sigma_\gamma \sigma_\epsilon$. But the composition $\tau\rho$ of a translation τ and a noncommuting rotation ρ couldn't be a translation ν , because then ρ would be the translation $\tau^{-1}\nu$, hence the identity, which commutes with τ . ♦

Exercises 16 to 18 don't cover the *whole* subject of commutativity. When do isometries of the various *other* types commute, and what are their compositions? Most of the combinations are too rare to warrant special treatment—except one: the composition of two rotations. In two dimensions its description is simple; see theorem 6.9.3. In three, that's not true unless the rotations commute. Exercise 19 investigates the situation.

Exercise 19. When do two rotations commute? What's their composition in that case and in general?

Solution. If rotations φ and χ about axes g and h commute, then $\chi = \varphi\chi\varphi^{-1}$, so $h = \varphi[h]$ by theorem 7.3.5. Thus $g = h$, or else $g \perp h$ and φ is the reflection across g . Reversing the roles of φ and χ yields $g = h$ or else $g \perp h$ and χ is the reflection across h . Thus rotations commute if and only if they have the same axis, or they're the half turns about perpendicular axes. In the first case $\varphi\chi$ is a rotation about the common axis. In the second, $\varphi\chi$ is the half turn about the line perpendicular to both axes.

The composition of two noncommuting rotations φ and χ is a screw. It could be a translation—for example, let φ and χ be the half turns about distinct parallel axes. It could be a rotation—for example, let φ and χ be 10° rotations (with the same sense) about distinct parallel axes. (Orient the axes so their positive rays fall on the same side of some plane.) It could be neither—for example, let φ and χ be the half turns about nonparallel lines g and h with a common perpendicular line. ♦

Exercises on equations

As in two-dimensional transformational geometry, the theory of isometries in three dimensions seems very different from the practice of computing with them. That's the effect of streamlining to present the essential details of the theory efficiently, in a form that's easy to remember. The exercises under this heading present the computational details. In the first two, closely related to two-dimensional exercise 6.11.25, you derive equations for plane reflections and half turns in terms of the equations of their mirrors and axes.

Exercise 20. Consider a point $P = \langle p_1, p_2, p_3 \rangle \neq O$ and the plane $\delta \perp \vec{OP}$ through O . Show that σ_δ has matrix

$$A = I - \frac{2}{P'P} K,$$

where $K = PP^t$. Now consider the plane $\varepsilon \perp \vec{OP}$ through a point Q . Show that $\sigma_\varepsilon : X \rightarrow X'$ has equation

$$X' = AX + \frac{2}{P^tP} KQ.$$

Solution. $X' = X + sP$ for some scalar s . The midpoint $M = \frac{1}{2}(X + X') = X + \frac{1}{2}sP$ lies on δ , so

$$0 = P^t(M - Q) = P^tX + \frac{1}{2}sP^tP - P^tQ$$

$$s = -\frac{2}{P^tP} P^t(X - Q)$$

$$\begin{aligned} X' &= X - \frac{2}{P^tP} (P^t(X - Q))P \\ &= X - \frac{2}{P^tP} (P^tX)P + \frac{2}{P^tP} (P^tQ)P. \end{aligned}$$

Now write the scalar multiple $(P^tX)P$ as the matrix product $P(P^tX) = (PP^t)X = KX$, and similarly $(P^tQ)P = KQ$. You get the expression for X' in the statement of the exercise. ♦

Exercise 21. Find an equation for the half turn $X \rightarrow X'$ about the line g with parametric equations $X = tP + Q$. *Suggestion:* Use the matrix $K = PP^t$ introduced in exercise 20.

Solution. Since $\vec{XX'} \perp g$ and the midpoint $M = \frac{1}{2}(X + X')$ lies on g , you can find t such that

$$M = Q + tP$$

$$0 = P^t(X - M) = P^t(X - Q - tP) = P^t(X - Q) - t(P^tP)$$

$$t = \frac{1}{P^tP} P^t(X - Q)$$

$$\begin{aligned} X' &= 2M - X = 2Q - X + 2tP \\ &= 2Q - X + \left(\frac{2}{P^tP} P^t(X - Q) \right) P \\ &= 2Q - \left(\frac{2}{P^tP} P^tQ \right) P - X + \left(\frac{2}{P^tP} P^tX \right) P \\ &= 2Q - \frac{2}{P^tP} (PP^t)Q - X + \frac{2}{P^tP} (PP^t)X \\ &= \left(-I + \frac{2}{P^tP} K \right) X + 2 \left(I - \frac{1}{P^tP} K \right) Q. \quad \diamond \end{aligned}$$

In the next two exercises, you start from geometric descriptions of certain isometries and find their equations. These involve very tedious computations. Use mathematical software wherever possible!

Exercise 22. Find equations for

- (a) the reflection across the plane ε with equation $x_1 + 2x_2 + 3x_3 = 4$;
- (b) the glide reflection with mirror ε and vector $\langle 3, -2, \frac{1}{9} \rangle$;
- (c) the half turn about the line through points $\langle 1, 1, 2 \rangle$ and $\langle 2, 1, 1 \rangle$.

Solution, Part 1. ε passes through $Q = \langle 1, 1, 1 \rangle$, and $\varepsilon \perp \vec{OP}$, where $P = \langle 1, 2, 3 \rangle$. Thus $P^t P = 1 \cdot 1 + 2 \cdot 2 + 3 \cdot 3 = 14$,

$$K = \begin{bmatrix} 1^2 & 1 \cdot 2 & 1 \cdot 3 \\ 2 \cdot 1 & 2^2 & 2 \cdot 3 \\ 3 \cdot 1 & 3 \cdot 2 & 3^2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$$

$$A = I - \frac{2}{14}K = \begin{bmatrix} \frac{6}{7} & -\frac{2}{7} & -\frac{3}{7} \\ -\frac{2}{7} & \frac{3}{7} & -\frac{6}{7} \\ -\frac{3}{7} & -\frac{6}{7} & -\frac{2}{7} \end{bmatrix}$$

$$\frac{1}{7}KQ = \frac{1}{7} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 6 \\ 12 \\ 18 \end{bmatrix}$$

$$X' = \begin{bmatrix} \frac{6}{7} & -\frac{2}{7} & -\frac{3}{7} \\ -\frac{2}{7} & \frac{3}{7} & -\frac{6}{7} \\ -\frac{3}{7} & -\frac{6}{7} & -\frac{2}{7} \end{bmatrix} X + \begin{bmatrix} \frac{3}{7} \\ \frac{12}{7} \\ \frac{18}{7} \end{bmatrix}.$$

Part 2. The glide reflection is the composition $\tau \sigma_\varepsilon$ of the reflection in part (1) and the translation τ with vector $\langle 3, -2, \frac{1}{9} \rangle$, so its equation is

$$X' = \begin{bmatrix} \frac{6}{7} & -\frac{2}{7} & -\frac{3}{7} \\ -\frac{2}{7} & \frac{3}{7} & -\frac{6}{7} \\ -\frac{3}{7} & -\frac{6}{7} & -\frac{2}{7} \end{bmatrix} X + \begin{bmatrix} \frac{3}{7} \\ \frac{12}{7} \\ \frac{18}{7} \end{bmatrix} + \begin{bmatrix} 3 \\ -2 \\ \frac{1}{9} \end{bmatrix} = \begin{bmatrix} \frac{6}{7} & -\frac{2}{7} & -\frac{3}{7} \\ -\frac{2}{7} & \frac{3}{7} & -\frac{6}{7} \\ -\frac{3}{7} & -\frac{6}{7} & -\frac{2}{7} \end{bmatrix} X + \begin{bmatrix} \frac{24}{7} \\ -\frac{2}{7} \\ \frac{269}{63} \end{bmatrix}$$

Part 3. Follow the solution of exercise 21. The equation is $X' = AX + B$, where

$$A = - \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}. \blacklozenge$$

Exercise 23. Consider the plane ε in exercise 22, its intersections P , Q , and R with the coordinate axes, and the line $g \perp \varepsilon$ through Q . Find equations for

- the rotation $\rho: X \rightarrow X'$ about g that maps $\vec{QR}$ to $\vec{QP}$, with coefficients accurate to four decimal places;
- the rotary reflection $\rho\sigma_\varepsilon$.

Suggestion: The equation has the form $X' = AX + B$ for some matrix A and some vector B . Find four points X , no three of which are coplanar, for which you know X' . From the four equations $X' = AX + B$ construct a system of twelve linear equations for the twelve entries of A and B , then solve it.

Solution, Part 1. Find P, Q, R and the points R^* and P^* on $\vec{QR}$ and $\vec{QP}$ such that $QR^* = QP^* = 1$. Then $\rho(Q) = Q$ and $\rho(R^*) = P^*$. Let $S = \langle 1, 2, 3 \rangle$ and $S^* = \|S\|^{-1}S$, so that $\|S^*\| = 1$, $U = Q + S$ lies on g and $\rho(U) = U$. Consider plane $\delta = P^*QS^*$ and let $T^* = (P^* - Q) \times S^*$, so that $\vec{OT}^* \perp \delta$ and $\|T^*\| = 1$. Let $V^* = \sigma_\delta(R)$ so that $\rho(P^*) = V^*$. By exercise 20, $V^* = JX + 2KQ$, where $J = I - 2K$ and $K = T^*(T^*)^t$. Solving for the entries of A and B the equations $X' = AX + B$ with $X = Q, R^*, U$, and P^* , you get

$$A = \begin{bmatrix} \frac{1}{14} + \frac{3}{70}\sqrt{65} & \frac{1}{7} - \frac{9}{91}\sqrt{65} & \frac{3}{14} + \frac{47}{910}\sqrt{65} \\ \frac{1}{7} + \frac{3}{35}\sqrt{65} & \frac{2}{7} + \frac{3}{91}\sqrt{65} & \frac{3}{7} - \frac{23}{455}\sqrt{65} \\ \frac{3}{14} - \frac{1}{14}\sqrt{65} & \frac{3}{7} + \frac{1}{91}\sqrt{65} & \frac{9}{14} + \frac{3}{182}\sqrt{65} \end{bmatrix}$$

$$\approx \begin{bmatrix} 0.4170 & -0.6545 & 0.6307 \\ 0.8339 & 0.5515 & 0.0210 \\ -0.3616 & 0.5172 & 0.7758 \end{bmatrix}$$

$$B = \begin{bmatrix} -\frac{2}{7} + \frac{18}{91}\sqrt{65} \\ \frac{10}{7} - \frac{6}{91}\sqrt{65} \\ -\frac{6}{7} - \frac{2}{91}\sqrt{65} \end{bmatrix} \approx \begin{bmatrix} 1.3090 \\ 0.8970 \\ -1.0343 \end{bmatrix}.$$

Part 2. Same as part (1), except $\rho(U) = 2Q - U$. You get

$$X' \approx \begin{bmatrix} 0.2741 & -0.9402 & 0.2021 \\ 0.5482 & -0.0199 & -0.8361 \\ -0.7902 & -0.3400 & -0.5100 \end{bmatrix} X + \begin{bmatrix} 1.8804 \\ 2.0399 \\ 0.6800 \end{bmatrix}. \blacklozenge$$

The final two exercises under this heading ask you to classify an isometry or similarity, given its equation. To *classify* an isometry,

- identify its class among those listed in the classification theorem;
- find an equation for its mirror—if it has one;
- find its vector—if it has one;
- find an equation for its axis—if it has one—and specify a vector to indicate a positive ray, then find a corresponding angle parameter.

To complete a classification, you may need to find fixpoints or fixed lines of an isometry $\varphi: X \rightarrow X' = AX + B$. For a fixpoint, you'll have to solve the linear system $X = X'$ —that is, $X = AX + B$, or $(I - A)X = B$. If $I - A$ is invertible, there's just one fixpoint X . Otherwise, reduce the system to an equivalent one for which you can identify the solutions readily. There are four cases. You might

- derive an *inconsistent* equation equivalent to $0 = 1$ —there's *no* fixpoint;
- reduce the system to three *trivial* equations, each equivalent to $0 = 0$ —*every* point is fixed;
- reduce it to two trivial equations and a nontrivial one that identifies a *plane* of fixpoints, with equation $c_1x_1 + c_2x_2 + c_3x_3 = d$;
- reduce it to one trivial equation and two equations

$$\begin{cases} x_i - c_i x_k = d_i \\ x_j - c_j x_k = d_j \end{cases},$$

where $\{i, j, k\} = \{1, 2, 3\}$ —you can convert those to parametric equations for a *line* of fixpoints:

$$\begin{cases} x_i = t c_i + d_i \\ x_j = t c_j + d_j \\ x_k = t \cdot 1 + 0 \end{cases}.$$

To find fixed lines g of an isometry φ , note that φ maps points $tP + Q$ with $P \neq O$ to points

$$A(tP + Q) + B = tP^* + Q',$$

where $P^* = AP$ and $Q' = AQ + B$. Thus φ maps the line $g \parallel \vec{OP}$ through Q onto a line $g' \parallel \vec{OP}^*$ through Q' . It follows that $g \parallel g'$ just in case P^* is a scalar multiple of P . But φ preserves length, so $g \parallel g'$ just in case $P^* = \pm P$. The first step in finding fixed lines is to find all vectors P such that $P^* = \pm P$. Solve the linear systems $AP = \pm P$ —that is, $(A \pm I)P = 0$. Second, find the points Q for which $g = g'$. For each P , both Q and Q' must lie on g , so

$$Q' - Q = AQ + B - Q = (A - I)Q + B = tP$$

for some t . You must solve the system

$$(I - A)Q + tP = B$$

of three linear equations in four unknowns: t and the components of Q .

That general procedure for finding fixed lines may be more involved than what you really need. For these classification exercises, your main fixed-line problem is probably to find the axis g of a screw $\varphi: X \rightarrow X' = AX + B$ that has no fixpoint. Here's a neater alternative for that special situation. Since φ affects points X on g like a translation with vector V ,

$$X' - X = V = (X')' - X',$$

$$AX + B - X = [A(AX + B) + B] - [AX + B]$$

$$(A^2 - 2A + I)X = (I - A)B$$

$$(I - A)^2X = (I - A)B.$$

You can find the points X on g by solving that singular linear system!

You can recognize a similarity $X \rightarrow AX + B$ with ratio r by computing $AA' = r^2I$; then classify the corresponding isometry $X \rightarrow r^{-1}AX + r^{-1}B$.

These exercises are stated in terms of exact fractions. To get *exact* solutions, you'll need mathematical software that operates with exact fractions.¹ You *won't* be able to carry out the computations by hand or by calculator with decimal arithmetic.

You *could* convert each fraction in an exercise to a decimal approximation, and use decimal arithmetic. Do some exercises that way, too! Carry as many decimal digits as your calculator permits. You'll have to make certain decisions in the classification procedure by testing whether some computed number $d = 0$. With approximate decimal arithmetic, you'll never get $d = 0$ *exactly*. So you'll have to decide how close to zero is close enough. In practice, that can be a critical decision. It's important enough in some areas of computer graphics that it's worth covering in more detail *there*. But *these* exercises provide exact fractions so you can clearly relate theory to computation without worrying about approximation errors that depend on your equipment.

¹ $X(\text{Plore})$ (Meredith 1993) is a good choice.

Exercise 24. For these matrices A verify that the equation $X' = AX$ defines an isometry or similarity $\varphi: X \rightarrow X'$, and classify it.

$$\begin{array}{ll}
 \text{(a)} \quad \begin{bmatrix} \frac{6}{7} & -\frac{2}{7} & -\frac{3}{7} \\ -\frac{2}{7} & \frac{3}{7} & -\frac{6}{7} \\ -\frac{3}{7} & -\frac{6}{7} & \frac{2}{7} \end{bmatrix} & \text{(b)} \quad \begin{bmatrix} \frac{12}{49} & \frac{31}{49} & -\frac{36}{49} \\ \frac{24}{49} & -\frac{36}{49} & -\frac{23}{49} \\ -\frac{41}{49} & -\frac{12}{49} & -\frac{24}{49} \end{bmatrix} \\
 \text{(c)} \quad \begin{bmatrix} -\frac{9518}{9555} & \frac{131}{1911} & \frac{526}{9555} \\ -\frac{142}{1911} & -\frac{1894}{1911} & -\frac{211}{1911} \\ -\frac{449}{9555} & \frac{218}{1911} & -\frac{9482}{9555} \end{bmatrix} & \text{(d)} \quad \begin{bmatrix} -6 & 2 & 3 \\ 2 & -3 & 6 \\ 3 & 6 & 2 \end{bmatrix}
 \end{array}$$

Solution, Part 1. Calculating $AA^t = AA = I$ and $\det A = -1$ shows that φ is a self-inverse odd isometry—that is, a point or plane reflection. Solving the singular linear system $X = X'$ yields the equivalent single equation $x_1 + 2x_2 + 3x_3 = 0$, so the fixpoints of φ are the points on the plane ε with that equation. That is, $\varphi = \sigma_\varepsilon$.

Part 2. Calculating $AA^t = I$ and $\det A = 1$ shows that φ is a rotation about a line through the origin. Solving the singular linear system $X = X'$ yields the equivalent system $x_1 + \frac{11}{7}x_3 = 0$, $x_2 + \frac{5}{7}x_3 = 0$, so the axis has equation $X = tR$, where $R = \langle -\frac{11}{7}, -\frac{5}{7}, 1 \rangle$. Let $\overrightarrow{OR}$ be its positive ray. The vector $Y = \langle 7, 0, 11 \rangle$ is perpendicular to the axis, $Y' = \langle -\frac{312}{49}, -\frac{85}{49}, -\frac{551}{49} \rangle$ and $\cos \angle YOY' = -\frac{97}{98}$. Since $Y \times Y'$ is a positive multiple of R , φ has angle parameter $\arccos(-\frac{97}{98}) \approx 171.8^\circ$.

Part 3. Calculating $AA^t = I$, $\det A = -1$, and $\det(A - I) \neq 0$ shows that φ is a rotary reflection with center O and no other fixpoints; it's not a reflection. $\varphi\varphi: X \rightarrow X'' = A^2X$ is a rotation with the same axis as φ and twice its angle parameter. Solving the singular linear system $X = X''$ yields the equivalent system $x_1 + \frac{11}{7}x_3 = 0$, $x_2 + \frac{5}{7}x_3 = 0$, so the axis has equation $X = tR$, where $R = \langle -\frac{11}{7}, -\frac{5}{7}, 1 \rangle$. Let $\overrightarrow{OR}$ be its positive ray. The mirror is the plane ε with equation $RX = 0$, that is, $11x_1 + 5x_2 - 7x_3 = 0$. The point $Y = \langle 7, 0, 11 \rangle$ lies in ε , $Y' = \langle -\frac{312}{49}, -\frac{85}{49}, -\frac{551}{49} \rangle$, and $\cos \angle YOY' = -\frac{97}{98}$. Since $Y \times Y'$ is a positive multiple of R , φ has angle parameter $\arccos(-\frac{97}{98}) \approx 171.8^\circ$.

Part 4. Calculating $AA^t = 49I$ shows that $\frac{1}{7}A$ is orthogonal. Moreover, $\det \frac{1}{7}A = 1$ and $(\frac{1}{7}A)^2 = I$, so the transformation $X \rightarrow \frac{1}{7}AX$ is the half turn about a line g through O . Its fixpoints are the solutions of $X = \frac{1}{7}AX$. Solving this singular linear system yields the equivalent system $x_1 - \frac{1}{3}x_3 = 0$, $x_2 - \frac{2}{3}x_3 = 0$, hence g has equation $X = t\langle \frac{1}{3}, \frac{2}{3}, 1 \rangle$. Thus φ is the homothetic 180° rotation about g with ratio 7. ♦

Exercise 25. For these matrices A and vectors B verify that the equation $X' = AX + B$ defines an isometry or similarity $\varphi: X \rightarrow X'$, and classify it.

- (a) Same A as exercise 24(a), $B = \langle 4/7, 8/7, 12/7 \rangle$;
 (b) Same A as exercise 24(a), $B = \langle 5, 6, 7 \rangle$;
 (c) Same A as exercise 24(b), $B = \langle 4/7, 8/7, 12/7 \rangle$;
 (d) Same A as exercise 24(c), $B = \langle 968/195, 88/39, -616/195 \rangle$;

$$(e) A = \begin{bmatrix} -\frac{9518}{9555} & -\frac{131}{1911} & \frac{526}{9555} \\ -\frac{142}{1911} & \frac{1894}{1911} & -\frac{211}{1911} \\ -\frac{449}{9555} & -\frac{218}{1911} & -\frac{9482}{9555} \end{bmatrix} \quad B = \langle 968/195, 88/39, -616/195 \rangle;$$

$$(f) A = \begin{bmatrix} -9518 & -655 & 526 \\ -710 & 9470 & -1055 \\ -449 & -1090 & -9482 \end{bmatrix} \quad B = \langle 968, 440, -616 \rangle.$$

Solution, Part 1. Calculating $AA^t = I$ and $\det A = -1$ shows that φ is a glide or rotary reflection. Moreover, $\varphi\varphi$ is the identity, so φ is a plane or point reflection. Solving the singular linear system $X = X'$ yields the equivalent single equation $x_1 + 2x_2 + 3x_3 = 4$, so the fixpoints of φ are the points on the plane ε with that equation. That is, $\varphi = \sigma_\varepsilon$.

Part 2. Calculating $AA^t = I$ and $\det A = -1$ shows that φ is a glide or rotary reflection. Moreover, $\varphi\varphi$ is the translation with vector $V = \langle 32/7, 8/7, -16/7 \rangle$, so φ is a glide reflection with vector $1/2 V$. The mirror of φ is that of the reflection $X \rightarrow AX + B - 1/2 V$. Solving the singular linear system $X = AX + B - 1/2 V$ yields the equivalent single equation $x_1 + 2x_2 + 3x_3 = 19$, so the mirror of φ is the plane with that equation.

Part 3. Calculating $AA^t = I$ and $\det A = 1$ shows that φ is a screw. Solving the singular linear system $X = X'$ yields the equivalent system $x_1 + 11/7 x_3 = 12/7$, $x_2 + 5/7 x_3 = 8/7$, so φ is a rotation whose axis g has equation $X = tR + S$, where $R = \langle -11/7, -5/7, 1 \rangle$ and $S = \langle 12/7, 8/7, 0 \rangle$. Let $\overrightarrow{OR}$ specify its positive ray. The vector $Y = \langle 7, 0, 11 \rangle$ is perpendicular to R , and S lies on g , so g is perpendicular to the line through points S and $Z = S + Y$, and $\cos \angle ZOZ' = -97/98$. Since $Y \times Y'$ is a positive multiple of R , φ has angle parameter $\arccos(-97/98) \approx 171.8^\circ$.

Part 4. Calculating $AA^t = I$ and $\det A = -1$ shows that φ is a glide or rotary reflection. Moreover, $\det(A - I) \neq 0$, so φ has exactly one fixpoint; it's a rotary reflection. Solving $X = X'$ yields the center $Z = \langle 484/195, 44/39, -308/195 \rangle$. The axis g consists of the fixpoints of $\varphi\varphi: X \rightarrow X''$. Inspecting the equation of $\varphi\varphi$ reveals that g passes through O . Solving the singular linear system $X = X''$ yields the equivalent system $x_1 + 11/7 x_3 = 0$, $x_2 + 5/7 x_3 = 0$, so g has equation $X = tR$, where $R =$

$\langle -\frac{11}{7}, -\frac{5}{7}, 1 \rangle$. Let $\vec{OR}$ be its positive ray. The mirror has equation $RX = RZ$, that is $11x_1 + 5x_2 - 7x_3 = 44$. Vector $Y = \langle 7, 0, 11 \rangle$ is perpendicular to R , so g is perpendicular to the line through points Z and $W = Z + Y$, and $\cos \angle WOW' = -\frac{97}{98}$. Since $Y \times Y'$ is a positive multiple of R , φ has angle parameter $\arccos(-\frac{97}{98}) \approx 171.8^\circ$.

Part 5. Calculating $AA^t = I$, $\det A = 1$, and $A \neq I$ shows that φ is a screw but not a translation. Moreover, $\det(A - I) = 0$ but $B \neq O$, hence there are no fixpoints— φ is not a rotation. A point X lies on its axis g just in case

$$\begin{aligned} X'' - X' &= X' - X \\ [A(AX + B) + B] - [AX + B] &= [AX + B] - X \\ (A^2 - 2A + I)X &= (I - A)B \\ (A - I)^2X &= -(A - I)B. \end{aligned}$$

Solving this singular linear system yields the equivalent system $x_1 - \frac{7}{11}x_3 = \frac{26656}{7639}$, $x_2 + \frac{195}{11}x_3 = \frac{-214032}{7639}$, so g has parametric equations $X = tR + Z$, where $R = \langle \frac{7}{11}, -\frac{195}{11}, 1 \rangle$ and $Z = \langle \frac{26656}{7639}, -\frac{214032}{7639}, 0 \rangle$. Point Z lies on the axis, so the vector of φ is $V = Z' - Z = \langle -\frac{616}{7639}, \frac{17160}{7639}, -\frac{968}{7639} \rangle$. Let V specify the positive direction. The vector $Y = \langle 11, 0, -7 \rangle$ is perpendicular to R , so g is perpendicular to the line through points Z and $W = Z + Y$, and $\cos \angle WOW' = -\frac{3817}{3822}$. Since $Y \times Y'$ is a positive multiple of R , φ has angle parameter $\arccos(-\frac{3817}{3822}) \approx 177.1^\circ$.

Part 6. Calculating $AA^t = 9555^2 I$ shows that $\frac{1}{9555}A$ is orthogonal and φ is a homothetic rotation with ratio $r = 9555$. Solving the linear system $X' = X$ yields the fixpoint $Z =$

$$\begin{aligned} &\langle \frac{44,188,543,916}{436,221,839,247}, -\frac{20,059,727,020}{436,221,839,247}, -\frac{28,119,319,492}{436,221,839,247} \rangle \\ &\approx \langle 0.101, -0.046, -0.064 \rangle. \end{aligned}$$

For points X on the axis, $X' - Z = r(X - Z)$. Solving this singular linear system yields the equivalent system

$$\begin{aligned} x_1 - \frac{7}{11}x_3 - \frac{12,996,160}{91,317,111} &= 0 \\ x_2 + \frac{195}{11}x_3 + \frac{108,555,280}{91,317,111} &= 0, \end{aligned}$$

hence the axis g has equation $X = tR + S$, where

$$\begin{aligned} R &= \langle \frac{7}{11}, -\frac{195}{11}, 1 \rangle \approx \langle 0.636, -17.727, 1 \rangle, \\ S &= \langle \frac{12,996,160}{91,317,111}, -\frac{108,555,280}{91,317,111}, 0 \rangle \approx \langle 0.142, -1.189, 0 \rangle. \end{aligned}$$

Let $\vec{OR}$ specify the positive ray. The vector $Y = \langle 11, 0, -7 \rangle$ is perpendicular to R , so g is perpendicular to the line through points Z and $W = Z + Y$, and $\cos \angle WOW' = -\frac{3817}{3822}$. Since $Y \times Y'$ is a positive multiple of R , φ has angle parameter $\arccos(-\frac{3817}{3822}) \approx 177.1^\circ$. ♦

Reflection calculus

Just as in two dimensions, geometers have developed approaches to three-dimensional geometry that rely very heavily on algebraic computations with compositions of self-inverse isometries—plane and point reflections and half turns. Because of the greater variety, these approaches are not so elegant as their plane counterparts. The *three-reflections theorems* provide basic tools in both two and three dimensions. The two-dimensional results are theorems 6.4.7, 6.5.6, and 6.7.2, and exercise 6.11.32. Their analogs haven't appeared yet in this more streamlined chapter. They're presented as exercises 26, 27, and 29. The intervening exercise 28 is an extension of lemma 7.5.2 that you'll need for exercise 29.

Exercise 26. Show that for any points P , Q , and R , the composition $\sigma_P \sigma_Q \sigma_R$ is the reflection across point $S = P - Q + R$, and $PQRS$ is a (possibly degenerate) parallelogram.

Solution. $PQRS$ is a (possibly degenerate) parallelogram because $P - Q = S - R$. Translations $\sigma_P \sigma_Q$ and $\sigma_S \sigma_R$ are equal by exercise 7, hence $\sigma_P \sigma_Q \sigma_R = \sigma_S$. ♦

Exercise 27. Show that the composition φ of the reflections across three planes is self-inverse in just three cases:

- (a) they're all perpendicular to a single line g , or
- (b) their intersection contains a line g , or
- (c) they're mutually perpendicular.

Also show that in case

- (a) φ is the reflection across a plane perpendicular to g , and
- (b) φ is the reflection across a plane through g , and
- (c) φ is the reflection across the intersection of the three planes.

Solution. First, suppose the three planes α , β , and γ are all perpendicular to g , and intersect g at points A , B , and C . Point $D = A - B + C$ lies on g ; let δ be the plane perpendicular to g through D . Then $\delta \parallel \alpha, \beta, \gamma$ and

$$\sigma_\alpha \sigma_\beta \sigma_\gamma = \sigma_\alpha \sigma_\beta \sigma_\gamma \sigma_\delta \sigma_\delta = \sigma_A \sigma_B \sigma_C \sigma_\delta = \sigma_D \sigma_\delta = \sigma_\delta.$$

Now suppose their intersection contains g . Select a point A on g ; consider the plane $\varepsilon \perp g$ through A and its intersections a , b , and c with α , β , and γ . The restrictions to ε of the reflections across α , β , and γ are the (two-dimensional) reflections of ε across a , b , and c . By theorem 7.2.1, for some line d through A in ε , the restriction of φ to ε coincides with the (two-dimensional) reflection σ_d of ε . Let δ be the plane that contains d and g . Since the even motion $\varphi \sigma_\delta$ fixes all points on g , it's

a rotation about g . But $\sigma_\delta(X) = \sigma_d(X) = \varphi(X)$ for all points X in ε , so $\varphi\sigma_\delta$ must be the identity, hence $\varphi = \sigma_\delta$.

Third, suppose they're mutually perpendicular and their intersection is a point P . Then $\sigma_\alpha\sigma_\beta\sigma_\gamma = \sigma_P$ by theorem 7.4.4.

Conversely, suppose φ is self-inverse. If $\beta = \gamma$, then the planes have a common perpendicular line or their intersection contains a line. So assume $\beta \neq \gamma$.

If $\beta \parallel \gamma$, then $\sigma_\beta\sigma_\gamma$ is a translation τ_V with $V \neq O$. By exercise 16, $\varphi = \sigma_\alpha\tau_V$ is a glide reflection, and $\tau_V = \sigma_\alpha\varphi$. Since it's self-inverse, $\varphi = \sigma_\delta$ for some plane δ , so $\sigma_\beta\sigma_\gamma = \tau_V = \sigma_\alpha\varphi = \sigma_\alpha\sigma_\delta$. Therefore, $\alpha, \beta, \gamma, \delta \perp OV$.

If $\beta \cap \gamma$ is a line g , then $\sigma_\beta\sigma_\gamma$ is a rotation ρ about g . Since $\varphi = \sigma_\alpha\rho$ is odd and self-inverse, it's a reflection across a plane δ or a point P . If $\varphi = \sigma_\delta$, then $\sigma_\beta\sigma_\gamma = \rho = \sigma_\alpha\varphi = \sigma_\alpha\sigma_\delta$, and α, β, γ , and δ all contain g .

Now suppose $\varphi = \sigma_P$. Select any point X on g ; then $\sigma_\alpha(X) = \sigma_\alpha\rho(X) = \varphi(X) = \sigma_P(X)$. If $\sigma_P(X) = X$, then $X = P$ and X lies on α , so P lies on α . Otherwise, the midpoint P of $\overline{X\sigma_P(X)}$ lies on α . In either case, P lies on α , so $\rho = \sigma_\alpha\varphi = \sigma_\alpha\sigma_P$, the half turn about the line perpendicular to α through P . Therefore, $\rho = \sigma_g$, $\alpha \perp g$, and P lies on g . Finally, $\beta, \gamma \perp \alpha$ because β and γ contain g , and $\beta \perp \gamma$ because $\sigma_\beta\sigma_\gamma = \rho$ is self-inverse. ♦

Exercise 28. According to lemma 7.5.2, for any lines b and c the composition $\sigma_b\sigma_c$ is a screw. Fill in the blanks and prove this extension of the lemma:

- $\sigma_b\sigma_c = \iota$ if and only if ...;
- it's a translation if and only if ...;
- it's a rotation if and only if ...;
- if it's not a translation, then its axis is the line that's perpendicular to b and c , and its angle parameter is ...;
- if it's not a rotation, then its vector is ...

Solution. $\varphi = \sigma_b\sigma_c$ is a screw, and $\varphi = \iota$ if and only if $b = c$.

If $b \parallel c$, select any plane $\varepsilon \perp b, c$; denote by B and C the intersections of ε with these lines. Then $\varphi = \sigma_b\sigma_c = \sigma_B\sigma_C$. By exercise 7 that's the translation with vector $2(B - C)$.

On the other hand, if φ is a translation τ_V , let $b' = \tau_{\frac{1}{2}V}[c]$. By the preceding paragraph, $\sigma_{b'}\sigma_c = \tau_V = \varphi$, so $\sigma_{b'} = \sigma_b$, hence $b' = b$, hence $b \parallel c$. Thus φ is a translation if and only if $b \parallel c$, and in that case its vector is $2(B - C)$, as described.

If $b \cap c$ is a point P , denote by β, γ the planes perpendicular to b, c through P . Then $\varphi = \sigma_b\sigma_P\sigma_P\sigma_c = \sigma_\beta\sigma_\gamma$, a rotation about the line $g \perp b$,

c . Its angle parameter is twice the measure of one of the dihedral angles formed by β and γ .

If b and c are skew, they have a unique common perpendicular line a that intersects them at points B and C . Denote by β and γ the planes determined by a with b and c . By the proof of lemma 7.5.2, φ is a screw with axis a and vector $2(B - C)$, and its angle parameter is twice the measure of one of the dihedral angles formed by β and γ . Thus φ is a rotation if and only if $b \cap c$ is a point, and in that case its axis and angle parameter are the line a and parameter θ as described.

Finally, if φ is not a translation, its axis and angle parameter have been described, and if it's not a rotation, its vector has been described. ♦

Exercise 29. Show that the composition φ of the half turns about three lines is self-inverse in just three cases:

- (a) the lines are mutually perpendicular, or
- (b) some pair of them is parallel, or
- (c) they have a common perpendicular line m .

Show also that in case

- (a) $\varphi = \iota$,
- (b) φ is the half turn about a line l ,
- (c) φ is the half turn about a line $l \perp m$.

Solution. Suppose lines g , h , and k are mutually perpendicular, meeting at point P , and denote by η and κ the planes through P perpendicular to g and h . Then $\sigma_g \sigma_h \sigma_k = \sigma_g \sigma_O \sigma_h \sigma_k = \sigma_\eta \sigma_\kappa \sigma_k = \sigma_k \sigma_k = \iota$.

Next, suppose $g \parallel h$, and consider any line k . By exercise 28, $\sigma_g \sigma_h$ is a translation τ with vector V , and $\sigma_g \sigma_h = \sigma_l \sigma_k$, where $l = \tau_{\frac{1}{2}V}[k]$. Thus $\sigma_g \sigma_h \sigma_k = \sigma_l$. If g were arbitrary but $h \parallel k$, then by the same argument, $\sigma_k \sigma_h \sigma_g = \sigma_l$ for some line l , and $\sigma_l = \sigma_l^{-1} = \sigma_g \sigma_h \sigma_k$. If h were arbitrary but $g \parallel k$, then by the same argument, $\sigma_g \sigma_k \sigma_h = \sigma_l$ for some line l , hence $\sigma_k \sigma_h = \sigma_g \sigma_l$, so $\sigma_h \sigma_k = \sigma_l \sigma_g$, and $\sigma_g \sigma_h \sigma_k = \sigma_g \sigma_l \sigma_g = \sigma_{\sigma_g(l)}$.

Third, suppose lines g , h , and k have a common perpendicular line m . By exercise 28, $\sigma_g \sigma_h$ is a screw, and there's a line $l \perp m$ such that $\sigma_g \sigma_h = \sigma_l \sigma_k$, hence $\sigma_g \sigma_h \sigma_k = \sigma_l$. Thus, in cases (a) to (c), φ is the identity or a half turn, as described.

Conversely, suppose $\sigma_g \sigma_h \sigma_k$ is self-inverse. If it's the identity, then $\sigma_g \sigma_h = \sigma_k$, and g, h, k are mutually perpendicular by exercise 28. If $\sigma_g \sigma_h \sigma_k = \sigma_l$ for some line l , then $\sigma_g \sigma_h = \sigma_k \sigma_l$. If $k \parallel l$, then by exercise 28, $\sigma_g \sigma_h = \sigma_k \sigma_l$ is a translation, hence $g \parallel h$. Otherwise, by exercise 28, $\sigma_g \sigma_h = \sigma_k \sigma_l$ is a screw but not a translation, and its axis m is perpendicular to g , h , k , and l . ♦

Exercise 30 is the three-dimensional analog of exercise 6.11.9.

Exercise 30. Show that every isometry φ is the composition of two self-inverse isometries. *Suggestion:* Consider separately the various types of isometry.

Solution. If φ is a glide reflection with mirror ε , then $\varphi = \sigma_\varepsilon \tau$ for some translation τ with vector parallel to ε . Therefore, $\tau = \sigma_P \sigma_Q$ for some points P and Q on ε , so $\varphi = \sigma_\varepsilon \sigma_P \sigma_Q = \sigma_h \sigma_Q$, where h is the line through P perpendicular to ε .

If φ is a rotary reflection with mirror ε , then $\varphi = \rho \sigma_\varepsilon$ for some rotation ρ with axis perpendicular to ε . Therefore, $\rho = \sigma_\gamma \sigma_\delta$ for some planes $\gamma, \delta \perp \varepsilon$, so $\varphi = \sigma_\gamma \sigma_\delta \sigma_\varepsilon = \sigma_\gamma \sigma_g$, where $g = \delta \cap \varepsilon$.

If φ is a screw with axis ε , then $\varphi = \rho \tau$ for some rotation ρ with axis g and some translation with vector parallel to g . Then $\rho = \sigma_\gamma \sigma_\delta$ for some planes γ and δ through g , and $\tau = \sigma_\varepsilon \sigma_\zeta$ for some planes $\varepsilon, \zeta \perp g$, so $\varphi = \sigma_\gamma \sigma_\delta \sigma_\varepsilon \sigma_\zeta = \sigma_\gamma \sigma_\varepsilon \sigma_\zeta = \sigma_h \sigma_k$, where h and k are the intersections of γ and δ with ε . ♦

Working in the 1920s and early 1930s on the two-dimensional reflection calculus—see that heading in section 6.11—Gerhard Thomsen evidently considered analogous three-dimensional questions. Thomsen's Ph.D. student H. Boldt developed that material into a full theory, published as his dissertation in 1934. It isn't as clean as its two-dimensional counterpart. That's understandable; three-dimensional theorems are generally more complicated. As you've seen in previous exercises on commuting reflections and on the three-reflections theorems, enlarging the class of nontrivial self-inverse isometries from two types to three brings minor complications to almost every elementary theorem. Consequently, even only slightly less elementary results become unwieldy when stated solely in terms of reflections and half turns. Exercises 31 and 32 are taken from Boldt 1934. They're closely related to the two-dimensional reflection calculus exercises in section 6.11.

Exercise 31. Describe the figure formed by points A, B, C , and G for which $\sigma_A \sigma_G \sigma_B \sigma_G \sigma_C \sigma_G = \iota$. Find an analogous result with points A, B, C, D , and G . *Suggestion:* Reason with vectors.

Solution. The first result is analogous to the two-dimensional exercise 6.11.33: G must be the centroid of $\triangle ABC$. Its proof is the same.

If A to D are collinear, then $\sigma_A \sigma_G \sigma_B \sigma_G \sigma_C \sigma_D \sigma_G = \iota$ just when the figure consists of four collinear points and their "average." If A to D are coplanar but not collinear, then the figure consists of these points and their "average." If A to D are noncoplanar, it's the tetrahedron $ABCD$ with its centroid G . ♦

Exercise 32, Part 1. Under what conditions on lines g and h is $\sigma_g \sigma_h \sigma_g \sigma_h = \iota$?

Part 2. Under what conditions on a line g and points P and Q is $\sigma_g \sigma_P \sigma_Q \sigma_g \sigma_Q \sigma_P = \iota$?

Part 3. Under what conditions on a line g and points P and Q is $\sigma_g \sigma_P \sigma_Q \sigma_g \sigma_P \sigma_Q = \iota$?

Part 4. Under what conditions on lines g and h do there exist points P and Q such that $\sigma_g \sigma_h = \sigma_P \sigma_Q$?

Part 5. Under what conditions on lines g and h do there exist points P and Q such that $\sigma_g \sigma_h \sigma_g \sigma_h = \sigma_P \sigma_Q$?

Suggestion: Exercise 28 is helpful.

Solution, Part 1. Since $\varphi = \sigma_g \sigma_h$ is even, this equation holds if and only if φ is the identity or a half turn. By exercise 28, that happens just when $g = h$ or $g \perp h$.

Part 2. The equation holds if and only if $\sigma_g \tau_{2(P-Q)} = \sigma_g \sigma_P \sigma_Q = \sigma_P \sigma_Q \sigma_g = \tau_{2(P-Q)} \sigma_g$. By theorem 7.5.1, this happens just when $P = Q$, or $P \neq Q$ and $g \parallel \vec{PQ}$.

Part 3. $\varphi = \sigma_g \sigma_P \sigma_Q$ is even. Moreover, $\varphi \neq \iota$ since σ_g can't be the translation $\sigma_P \sigma_Q$. Thus the equation holds if and only if $\varphi = \sigma_h$ for some line h , that is $\sigma_g \sigma_h = \sigma_P \sigma_Q$. By exercise 28, $\sigma_g \sigma_h$ is a translation just when $g \parallel h$, and in that case the translation vector $2(P - Q)$ is parallel to a line perpendicular to g and h . Therefore the equation holds just when $P = Q$, or $P \neq Q$ and $\vec{PQ}$ is perpendicular to some line parallel to g .

Part 4. By the argument for part (3), P and Q exist just when $g \parallel h$.

Part 5. $\varphi = \sigma_g \sigma_h$ is a screw and $\sigma_P \sigma_Q$ is a translation, so P and Q exist if and only if φ has angle parameter 0° or 180° . By exercise 28, that happens just when $g = h$, or g is parallel to a line perpendicular to h . ♦

Section

8.5

Solutions to exercises on

Symmetry

Concepts

- Examples of functional and ornamental symmetry
- Subgroup relationships among symmetry groups
- Interior of a simple closed polygon
- Triangulation
- Convex polygons
- Involutions
- Generating sets of groups
- Generators of translation groups
- Polyhedral models
- Schlegel diagrams
- Inradii and circumradii of regular polyhedra
- Interior of a polyhedron

This section contains forty-one exercises related to earlier parts of the chapter. Their chief goal is to introduce the detailed study of examples of the types of symmetry considered here. In eleven of these exercises you'll examine functional and ornamental designs incorporating polygonal symmetry,

friezes, wallpaper patterns, and polyhedral symmetry. You'll determine their symmetry groups and see how they're related. In five more exercises you'll search for further examples of symmetric design. Perhaps you can challenge other readers to classify these and find even more interesting ones. A few exercises complete some discussions in earlier sections. But there's no effort here to introduce new topics. To extend the scope of the inquiries in this text, go to the literature. The references in earlier sections, particularly those in the chapter introduction, should point the way. *Bon voyage!*

General exercises

The first two exercises encourage you to become alert to examples of symmetry all around you. When you discover a new example, you should consider whether the symmetry is functional or ornamental, and what other types of symmetry could have served the design just as well.

Exercise 1. Figure 8.5.1 shows some household objects. Describe their symmetries. Is symmetry necessary for these designs? Could the designers have used other symmetry types?

Solution. Not available. ♦

Exercise 2. Find and describe more examples of household symmetry, comparable to those in figure 8.5.1.

Solution. Not available. ♦

Exercises on polygons and polygonal symmetry

In exercises 3 to 5 you'll continue exploring examples of symmetry, finding and analyzing functional and ornamental polygonal designs. After that, two exercises remind you of the role of matrix algebra in this subject, and study the relationships between various polygonal symmetry groups. The last two exercises investigate the notion of regular polygon a little more deeply than section 8.1, and make the connection between the notion of simple closed polygon used here, and that of polygonal region introduced in chapter 3.

Exercise 3. Identify the symmetry groups of the ornaments in figure 8.5.2, from the San Ildefonso, Mimbres, Mescalero Apache, Cochiti, and Pima cultures of the southwestern United States.

Solution. (a) $\dots \mathcal{D}_2$
 (b) $\dots \mathcal{C}_8$ (c) $\dots \mathcal{D}_4$
 (d) $\dots \mathcal{C}_2$ (e) $\dots \mathcal{C}_6$. ♦



Figure 8.5.1
Household symmetry
examples¹

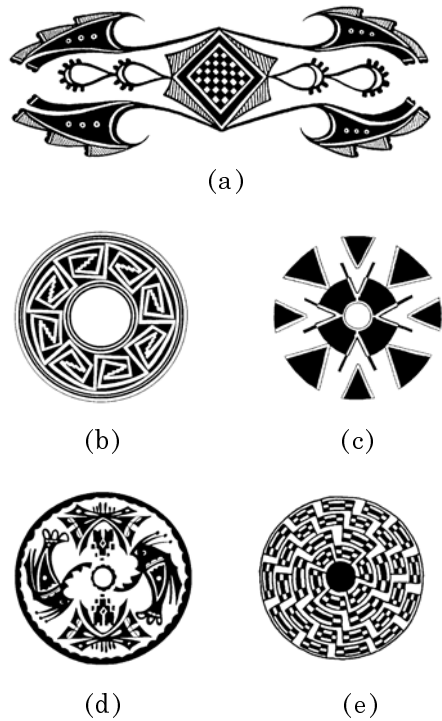


Figure 8.5.2
Five symmetric
designs²

Exercise 4. Classify uppercase Latin and Greek letters according to their symmetry groups. Use the most symmetric typefaces you can find. How about letters in other alphabets?

Solution. AMTUVWΛΠΨΩ BCDEKΣ FGPR HIΘΞΦ JQΓ
L NSZ O × ΥΔ

You could also argue that all letters with exactly one line of symmetry should be classified together. ♦

Exercise 5. Survey functional and ornamental symmetric polygons. Find examples of designs or ornaments with $\mathcal{C}_n$ or $\mathcal{D}_n$ symmetry groups, for

¹ Photograph by the author.
² From Kate 1997.

$n = 2, 3, 4, \dots$. Go up to as large an n as you can. Consider these categories or others you might think of:

- (a) Household objects, as in figure 8.5.1.
- (b) Coins, medals, medallions. Find familiar examples with seven or eleven vertices.
- (c) Nuts, bolts, and other tools and machinery. What kind of nut has five edges?
- (d) Logos, as in figure 8.5.3.
- (e) Road signs. For example, the Kansas highway sign in figure 8.5.4.
- (f) Architecture: rooms and buildings.
- (g) Ornament. Analyze the plate by contemporary San Ildefonso artist Tse-pe in figure 8.5.5. (It's rich brown, with smooth and rough textures and a turquoise pebble.) Which symmetry groups are employed in the two-century-old jar in figure 8.5.6 from nearby Tesuque Pueblo? Find other examples from various cultures.
- (h) Flowers, fruits, vegetables.

Solution. (b) Many police emblems have $\mathcal{D}_7$ symmetry. The Susan B. Anthony U.S. dollar and Canadian “loonie” dollar each have eleven edges.

(c) Fire hydrants have rare pentagonal nuts so that only firemen can open them easily.

(e) $\mathcal{C}_8$, ignoring numerals and rivets.

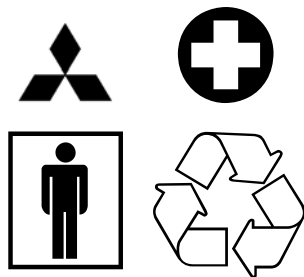


Figure 8.5.3
Symmetric logos



Figure 8.5.4
Kansas highway sign³

³ Photograph by the author.



Figure 8.5.5
Plate by Tse-pe⁴



Figure 8.5.6
Tesuque jar⁵

(f) The plate is basically a $\mathcal{D}_3$ design. Tse-pe added swirling bear motifs to suggest $\mathcal{C}_3$, then placed different designs inside them to break the symmetry entirely. The $\cdots\text{BBBB}\cdots$ frieze around the jar top is wound into a $\mathcal{C}_{14}$ or perhaps $\mathcal{C}_{15}$ pattern. The left-hand design appears to have $\mathcal{C}_2$ symmetry. The border of the right-hand pattern is $\mathcal{C}_{16}$. The flower inside it is $\mathcal{D}_7$. ♦

Exercise 6. Find the matrices for the elements of a $\mathcal{D}_6$ group with center at the origin.

Solution. $\begin{bmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{bmatrix}$ and $\begin{bmatrix} \cos n\theta & \sin n\theta \\ \sin n\theta & -\cos n\theta \end{bmatrix}$ for $\theta = 60^\circ$ and

$n = 0$ to 5, by the discussion in section 6.8. ♦

Exercise 7, Part 1. For which m and n is a $\mathcal{C}_m$ group a subgroup of a $\mathcal{C}_n$ group? In that case, how many subgroups are isomorphic to it? Are they all conjugate?

Part 2. Same as part 1 for $\mathcal{C}_m$ and $\mathcal{D}_n$.

Part 3. Same as part 1 for $\mathcal{D}_m$ and $\mathcal{D}_n$.

Part 4. How many subgroups has a $\mathcal{D}_6$ or $\mathcal{D}_{13}$ group?

⁴ Dillingham 1994, 243.

⁵ Frank and Harlow 1974, plate 39.

Solution. A $\mathcal{C}_m$ or $\mathcal{D}_m$ group is a subgroup of a $\mathcal{C}_n$ or $\mathcal{D}_n$ group if and only if m divides n . In that case there's just one $\mathcal{C}_m$ subgroup, but n/m $\mathcal{D}_m$ subgroups. $\mathcal{D}_6$ and $\mathcal{D}_{13}$ groups have sixteen subgroups. ♦

Exercise 8. Suppose the n vertices of a simple closed polygon Π fall in order on a circle.

Part 1. Show that Π is regular if its edges are all congruent. What if the vertices are not in order, or don't fall on a circle?

Part 2. Show that Π is regular if its vertex angles are all congruent and n is odd. What if n is even?

Solution, Part 1. Use the SSS congruence principle. If the vertices lie on a circle, they must fall in order. If they don't lie on a circle, the polygon needn't be regular—consider a nonsquare rhombus.

Part 2. If Π is a triangle, the result is obvious. Otherwise, consider successive vertices A to D and the center O of the circle. Theorem 3.13.6, with some arithmetic, implies $\angle AOB \cong \angle COD$, hence $AB = CD$. This pattern persists around the circle, and implies that all edges are congruent if n is odd. If n is even, Π needn't be regular—consider a nonsquare rectangle. ♦

Exercise 9, Part 1. Define the notion *interior* of a simple closed polygon Π . *Suggestion:* This is harder than it seems. First, show how to find some point E that clearly should be called exterior. The interior points X are those for which the segment $\overline{EX}$ crosses Π an odd number of times. Now you need to define the word *crosses*.

Part 2. Let $\bar{\Pi}$ be the union of a simple closed polygon Π and its interior. Devise instructions—a *triangulation* algorithm—for dividing $\bar{\Pi}$ into triangular regions that intersect only along edges. This will show that $\bar{\Pi}$ is a polygonal region as defined in section 3.8. *Suggestion:* This, too, is harder than it seems, and you probably won't find a completely reliable algorithm. But you'll learn by proposing one and defending it against or accepting others' objections. If Π is convex—that is, two vertices never lie on different sides of any edge line—the problem is simpler. You'll find it discussed in books on computational geometry; for example, see Preparata and Shamos 1985.

Solution, Part 1. A polygon lies inside some circle; choose E outside that. In defining *cross* you have to handle cases where $\overline{EX}$ contains a vertex or edge of Π . Or, you could choose E so that doesn't happen.

Part 2. Not available. ♦

Exercises on frieze groups

Exercises 10 to 12 continue your exploration of functional and ornamental symmetry examples. After those, two exercises clarify the relationship

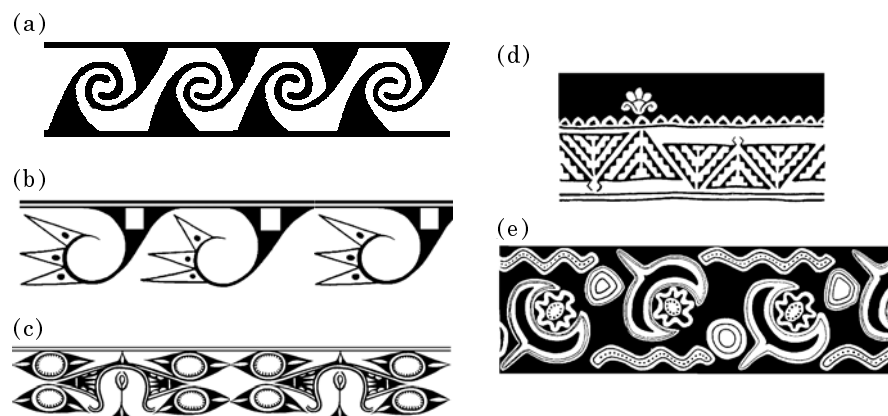


Figure 8.5.7
Five American friezes⁶

between conjugacy and isomorphy of frieze groups, and introduce the use of generators for describing isometry groups. Exercise 15 asks you to devise a frieze classification tree more compact than the one in section 8.2.

Exercise 10. Identify the symmetry groups of the friezes in figure 8.5.7 from various cultures of the southern and southwestern United States.

Solution. (a) $\cdots \text{NNNN} \cdots$ (c) $\cdots \text{AAAA} \cdots$
(b) $\cdots \text{FFFF} \cdots$ (d) $\cdots \text{MWMW} \cdots$
(e) $\cdots \text{pbpb} \cdots$ ♦

Exercise 11. Classify friezes constructed from repeated letters of the uppercase Latin and Greek alphabets, using the typefaces you chose in exercise 4. Which friezes are classified the same, even though the letters are classified differently? Which frieze groups fail to occur? Can you construct examples of those using repeated *pairs* of letters? Which pairs?

Solution.

$\{ \cdots \text{AAAA} \cdots, \cdots \text{MMMM} \cdots, \cdots \text{TTTT} \cdots, \cdots \text{UUUU} \cdots, \cdots \text{VVVV} \cdots, \\ \cdots \text{WWWW} \cdots, \cdots \text{YYYY} \cdots, \cdots \text{DDDD} \cdots, \cdots \text{LLLL} \cdots, \cdots \text{PPPP} \cdots, \\ \cdots \text{QQQQ} \cdots, \cdots \text{RRRR} \cdots \}$
 $\{ \cdots \text{BBBB} \cdots, \cdots \text{CCCC} \cdots, \cdots \text{DDDD} \cdots, \cdots \text{EEEE} \cdots, \cdots \text{KKKK} \cdots, \\ \cdots \text{SSSS} \cdots \}$

⁶ From Kate 1997. Figure 8.5.7(e) has been altered slightly to enhance its symmetry.

$$\begin{aligned} &\{\cdots \text{HHHH} \cdots, \cdots \text{I I I I} \cdots, \cdots \text{O O O O} \cdots, \cdots \text{x x x x} \cdots, \cdots \text{O O O O} \cdots, \\ &\quad \cdots \text{E E E E} \cdots, \cdots \Phi \Phi \Phi \Phi \cdots\} \\ &\{\cdots \text{FFFF} \cdots, \cdots \text{G G G G} \cdots, \cdots \text{J J J J} \cdots, \cdots \text{L L L L} \cdots, \cdots \text{P P P P} \cdots, \\ &\quad \cdots \text{Q Q Q Q} \cdots, \cdots \text{R R R R} \cdots, \cdots \text{r r r r} \cdots\} \\ &\{\cdots \text{NNNN} \cdots, \cdots \text{S S S S} \cdots, \cdots \text{Z Z Z Z} \cdots\} \end{aligned}$$

In the $\cdots \text{mwmw} \cdots$ frieze class, there's $\cdots \text{VAVA} \cdots$. You could also change the U typeface to Π and pair it with Π to get $\cdots \Pi \Pi \Pi \Pi \cdots$. Finally, in the $\cdots \text{pbpb} \cdots$ class is $\cdots \text{L r L r} \cdots$. ♦

Exercise 12. Survey functional and ornamental friezes. Consider these categories and others you may think of:

- (a) border ornaments on clothing and flat art works;
- (b) pottery decoration;
- (c) architectural ornaments, fences, balusters, and gratings;
- (d) tools.

Find friezes representing all seven classes.

Solution. Not available.

Exercise 13. In section 8.2, frieze groups are partitioned into conjugacy classes. But frieze groups may be isomorphic without being conjugate. Divide them into isomorphism classes. *Suggestion:* To prove that two of these groups are nonisomorphic, concentrate on their *involutions*—self-inverse elements different from the identity.

Solution. The $\cdots \text{AAAA} \cdots$, $\cdots \text{NNNN} \cdots$, and $\cdots \text{mwmw} \cdots$ groups are isomorphic; they contain many involutions, no two of which commute. The $\cdots \text{BBBB} \cdots$ groups contain exactly one involution. The $\cdots \text{FFFF} \cdots$ and $\cdots \text{pbpb} \cdots$ groups are isomorphic; they contain no involutions. The $\cdots \text{HHHH} \cdots$ groups contain distinct involutions that commute. Therefore there are exactly four isomorphism classes:

$$\cdots \text{AAAA} \cdots \quad \cdots \text{BBBB} \cdots \quad \cdots \text{FFFF} \cdots \quad \cdots \text{HHHH} \cdots.$$

To obtain an isomorphism from an $\cdots \text{AAAA} \cdots$ group to an $\cdots \text{NNNN} \cdots$ group, map reflections to half turns. To obtain an isomorphism from an $\cdots \text{mwmw} \cdots$ group to an $\cdots \text{NNNN} \cdots$ group or from a $\cdots \text{pbpb} \cdots$ group to an $\cdots \text{FFFF} \cdots$ group, map glide reflections to translations with the same vector. ♦

Exercise 14. A subset $\mathcal{P}$ of a group $\mathcal{G}$ *generates* it if no proper subgroup of $\mathcal{G}$ contains $\mathcal{P}$. For example, a finite cyclic isometry group is generated by a set consisting of a single rotation; the corresponding dihedral group is generated by a set consisting of that rotation and a single reflection. Find

generating sets as small as possible for each of the conjugacy classes of frieze groups. Which have two-element generating sets? Which have generating sets consisting solely of involutions?

Solution. Consider

- a nontrivial translation τ ,
- distinct lines $v, w \perp \tau$ and $h \parallel \tau$,
- distinct points P, Q on h but not on v ,
- the glide reflection γ with axis h and vector $\frac{1}{2}\vec{\tau}$.

Here some generating sets. Only the indicated groups have two-element or involutory generating sets.

<i>Group</i>	<i>Generating set</i>	<i>Generating set of involutions</i>
$\cdots \text{AAAA} \cdots$	τ, σ_v	σ_v, σ_w
$\cdots \text{BBBB} \cdots$	τ, σ_h	
$\cdots \text{FFFF} \cdots$	τ	
$\cdots \text{HHHH} \cdots$	τ, σ_v, σ_h	$\sigma_v, \sigma_h, \sigma_P$
$\cdots \text{NNNN} \cdots$	τ, σ_P	σ_P, σ_Q
$\cdots \text{MWMW} \cdots$	γ, σ_P	
$\cdots \text{pbpb} \cdots$	γ	$\blacklozenge$

Exercise 15. Since there are seven conjugacy classes of friezes and $7 < 2^3$, it must be possible to classify a frieze with at most three successive questions. Design a tree with that property; try to keep the questions as simple as possible.⁷

Solution. See figure 8.5S.1. $\blacklozenge$

Exercises on wallpaper groups

These five exercises on wallpaper groups continue the emphasis of the previous exercises. You'll survey and analyze more examples of symmetry and clarify the use of generators for describing these groups. Exercise 20 investigates the subgroup relationships between wallpaper groups. It shows why you can't use this concept to describe why one wallpaper pattern might seem more symmetric than another, and may give you an idea why it's sometimes so hard to classify wallpaper patterns.

⁷ Designing efficient decision trees is a challenge. For example, the key to central Californian marine shelled gastropods in Light et al. 1961, 243–259, organizes about ninety species. Unlike the frieze tree, that one doesn't ask the same question of each branch at each level, so it can be more efficient. But its depth is twelve, considerably greater than the theoretically optimal $7 = \lceil \log_2 90 \rceil$. You may have to ask as many as twelve questions to classify a particular specimen—for example, *Acmaea limatula*.

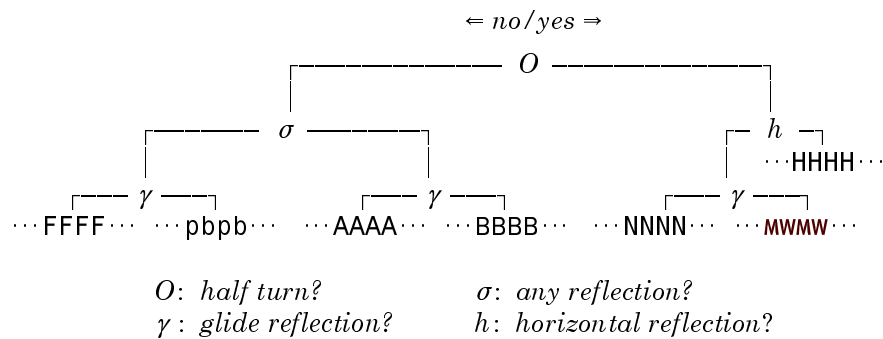


Figure 8.5S.1
Exercise 15 solution⁸

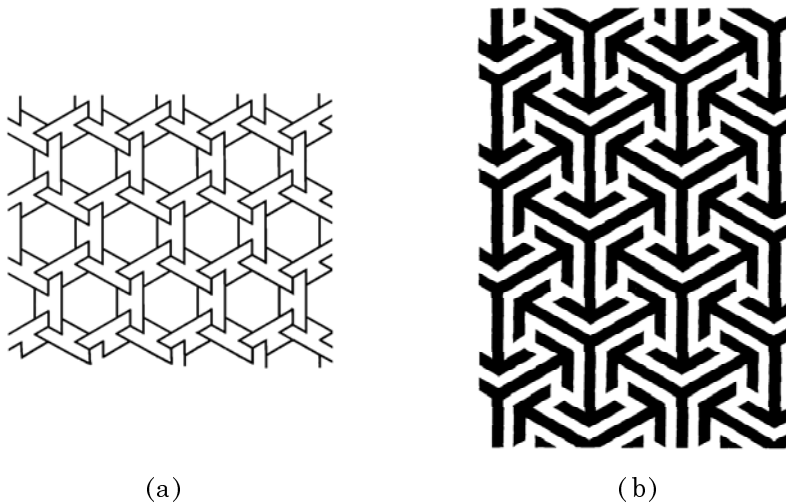


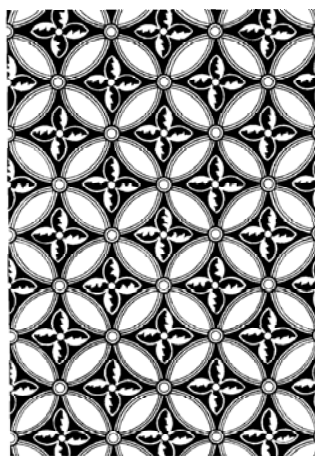
Figure 8.5.8 (part 1 of 3)
Example wallpaper patterns⁹

⁸ Due to Martin 1982, privately distributed solutions manual, 24.

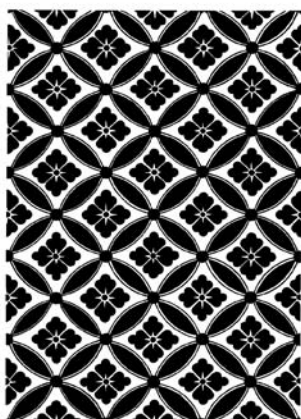
⁹ (a) is an Iranian design from Wade 1982. (b) and (c) are Arab ornaments and (d) to (j), Japanese kimono and obi designs, from Audsley and Audsley [1882] 1968. The latter authors and their contemporaries termed these *diaper* ornaments, not because of their functionality, but because *diaper* stems from Greek words meaning *all white*. These patterns are designed to be printed on plain white cloth!



(c)



(d)



(e)



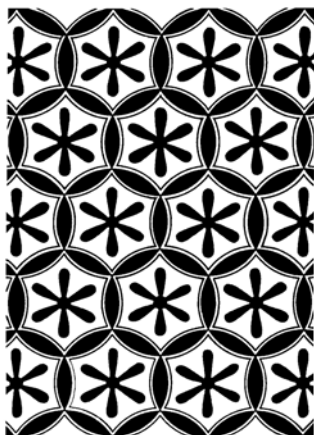
(f)

Figure 8.5.8 (part 2 of 3)

Example wallpaper patterns

Exercise 16. Identify the symmetry groups of the ten wallpaper patterns in figure 8.5.8.

Solution. (a) $p31m$ (b) $p3m1$ (c) $p6$ (d) $p4$ (e) $p4m$
 (f) pm (g) $p6m$ (h) $p1$ (i) cmm (j) pgg ♦



(g)



(h)



(i)



(j)

Figure 8.5.8 (part 3 of 3)

Example wallpaper patterns

Exercise 17. Construct examples of all classes of wallpaper patterns from repeated letters of the upper- or lowercase Latin, Greek, or Cyrillic alphabets.

Use the typefaces you chose in exercise 4. When possible, use single-letter motifs. You may vary their spacing to produce different symmetries. Use motifs based on two or more letters if required. But use letter **O** only as absolutely necessary. (You should have to resort to that in only three cases.)

<i>Solution.</i>	<i>p1:</i>	FFF FFF FFF	<i>cm:</i>	A A A A A A A A A A	<i>pm:</i>	AAA AAA AAA
<i>pg:</i>	A A A A A A A A A A A A	<i>p6:</i>	Construct the figure 8.3.2 motif from o's.		<i>p6m:</i>	o o o o o o o o o o
<i>p3:</i>	Construct the figure 8.3.19 motif from o's.	<i>p3m1:</i>			<i>p31m:</i>	
<i>p4:</i>		<i>p4m:</i>			<i>p4g:</i>	
<i>p2:</i>	NNN NNN NNN	<i>cmm:</i>			<i>pmm:</i>	
<i>pmg:</i>	NINI NINI NINI NINI	<i>pgg:</i>				

Exercise 18. Use this book's references, your library, museums, and commercial shops to find more examples of wallpaper patterns. In particular, find examples of the classes not represented in exercise 16. Challenge other readers to classify them.

Solution. Not available. ♦

Exercise 19. Suppose φ and χ are the fundamental translations of a wallpaper group $\mathcal{W}$, and consider any integers m_1 , m_2 , n_1 , and n_2 . Show that $\varphi^{m_1}\chi^{n_1}$ and $\varphi^{m_2}\chi^{n_2}$ generate the group of all translations in $\mathcal{W}$ if and only if

$$\det \begin{bmatrix} m_1 & n_1 \\ m_2 & n_2 \end{bmatrix} \neq 0.$$

Solution. These translations generate the translation group if and only if for all integers m and n there exist integers p and q such that

$$\varphi^m \chi^n = (\varphi^{m_1} \chi^{n_1})^p (\varphi^{m_2} \chi^{n_2})^q;$$

that is, if and only if for all m and n there exist p and q such that

$$m\vec{\varphi} + n\vec{\chi} = (pm_1 + qm_2)\vec{\varphi} + (pn_1 + qn_2)\vec{\chi}.$$

Since $\vec{\varphi}$ and $\vec{\chi}$ aren't parallel, this equation is equivalent to the pair of equations $m = pm_1 + qm_2$ and $n = pn_1 + qn_2$. The theory of simultaneous linear equations yields the desired result. ♦

Exercise 20. Consider $p3m1$ and $p1m3$ groups $\mathcal{G}$ and $\mathcal{H}$. Show that each is isomorphic to a subgroup of the other. Find another pair of wallpaper group classes with this property. *Suggestion:* Find a $p3m1$ pattern inside a $p1m3$ pattern, and vice versa.

Solution. Delete two of every three motifs from the $p3m1$ pattern P in figure 8.3.20 to get the $p1m3$ pattern Q in figure 8.5S.2. Since you can reconstruct P from Q , every symmetry of Q is a symmetry of P . That is, the $p1m3$ symmetry group $\mathcal{G}$ of Q is a subgroup of the $p3m1$ group $\mathcal{H}$ of P . The same process applied to figure 8.3.21 yields figure 8.5S.3 and shows that a $p3m1$ group is a subgroup of a $p1m3$ group. It follows that $\mathcal{H}$ is isomorphic to a subgroup of $\mathcal{G}$.

Only two other pairs of classes of wallpaper groups have this property: pm, cm and pmm, cmm . It's psychologically easier to detect analogous relationships between figures 8.3.11 and 8.3.12 (pm and cm) and between figures 8.3.29 and 8.3.28 (pmm and cmm).¹⁰ ♦

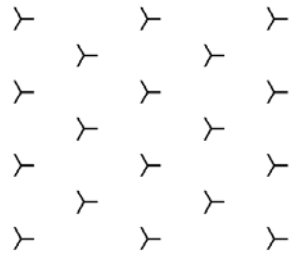


Figure 8.5S.2 Extracted
from figure 8.3.20

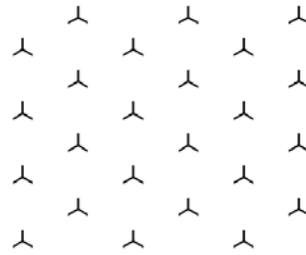


Figure 8.5S.3 Extracted
from figure 8.3.21

¹⁰ See Coxeter and Moser 1965, section 4.6.

At first thought, it seems reasonable to call a figure G “at most as symmetric as” a figure H if the symmetry group $\mathcal{G}$ of G is isomorphic to a subgroup of the symmetry group $\mathcal{H}$ of H . But exercise 8.5.20 belies that; it’s possible that each group is isomorphic to a subgroup of the other, yet the two figures have different symmetry properties. You can show that this can’t happen for cyclic and dihedral groups, or for the frieze groups. Perhaps this phenomenon accounts for the difficulty we sometimes have in distinguishing the symmetry properties of wallpaper patterns.

Exercises on polyhedra

Models are the best tools for studying polyhedra. Beautiful, expensive commercial products are available.¹¹ Since they’re either ready-made or require assembly and fastening with glue, they don’t offer much experience in construction. You don’t get the insight that results from trying several times to fit the faces together. However, there’s a simple way to construct models with inexpensive materials.

To make a regular tetrahedron, for example, first decide on the edge length. 3 inches works well.¹² You need a piece of stiff cardboard—*railroad board* is best—large enough that you can cut from it five equilateral triangles with edges somewhat longer than 3 inches. Cut out one equilateral triangle T as in figure 8.5.9 to use as a template. Use right triangle trigonometry to compute the T edge length $3 + \frac{1}{2}\sqrt{3} \approx 3.87$ inches, and to locate points A , B , and C near the vertices $\frac{1}{4}$ inch from the edges. Puncture T at

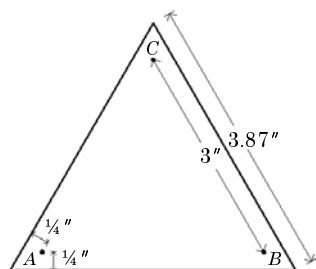


Figure 8.5.9 Equilateral triangular face template

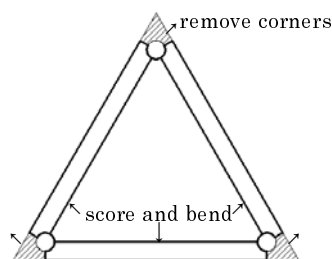


Figure 8.5.10 Equilateral triangular face for models

¹¹ See Symmetrics [no date], Cuisenaire Dale Seymour Publications 1998, and Key Curriculum Press 1997–1998, or more recent editions of these catalogs.

¹² Stewart (1980) recommended these specifications. The edge length, tab width, hole diameter, rubber-band dimension, and rigidity of the railroad board must be compatible. Different specifications could produce a model that sags or buckles.

those spots with holes just large enough to accommodate a pencil point. Use the punctured T as a template to make four copies T' with pencil marks at A , B , and C . With straightedge and stylus score the edges of $\triangle ABC$. With pencil mark the segments S from those points perpendicular to the nearest edges. Use a hand punch to make circular holes with diameter $\frac{1}{4}$ inch centered at A , B , and C . Make scissor cuts along the remaining portions of segments S , removing the corners from triangles T' . The resulting face components resemble figure 8.5.10. Bend the rectangular flaps away from you along the scores. Use them as exterior tabs with $1\frac{1}{4}$ -inch rubber bands to attach faces to each other. You've now assembled a regular tetrahedron with exterior tabs. The tabs make the model less than ideal in appearance, but that's an acceptable trade-off for ease of manufacture, disassembly, and reuse.

You should make a stock of triangular, square, pentagonal, hexagonal, ..., tabbed faces, with the same edge length, and acquire a supply of rubber bands of the appropriate type. When you're finished with a particular model, you can disassemble it for easy storage, or for reuse of its components.¹³

Exercise 21. Make and thoroughly inspect models of the polyhedra considered in section 8.4.

Solution. Not available. ♦

Exercise 22. Describe the following intersections, including all possible cases:

- (a) of a regular tetrahedron with a plane parallel to two opposite edges,
- (b) of a cube with a plane perpendicular to the line joining two opposite vertices,
- (c) of a regular octahedron with a plane parallel to two opposite faces,
- (d) of a regular dodecahedron with a plane parallel to two opposite faces,
- (e) of a regular icosahedron with a plane perpendicular to the line joining two opposite vertices.

Solution. See figure 8.5S.4. ♦

A *Schlegel* diagram for a convex polyhedron is a perspective image of its edges on the plane of one face F , with exterior focal point so close to F that the images of all the other edges fall within F . For example, figure 8.5.11 is a Schlegel diagram of a cube, with focus near the center of a face F . The outside square is the image of F ; the inside one, that of the opposite face. Evidently, a Schlegel diagram for a polyhedron with n faces divides

¹³ Stretched rubber bands may deteriorate if the model is stored assembled.

(a)	—						
(b, c)	·	Δ	truncated Δ	regular hexagon	truncated ∇	∇	·
(d, e)	·	regular pentagon	truncated regular decagon	regular decagon	truncated regular decagon (inverted)	regular pentagon (inverted)	·

Figure 8.5S.4
Exercise 22 solution

the plane into $n - 1$ polygonal regions and a surrounding unbounded region representing the face nearest the focus.

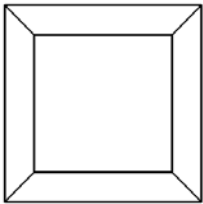
Exercise 23. Why is the use of Schlegel diagrams restricted to *convex* polyhedra? Make Schlegel diagrams for the regular polyhedra, cuboctahedron, rhombic dodecahedron, equilateral triangular and pentagonal prisms and dipyrramids, square antiprism, and the deltahedrally stellated triangular prism.

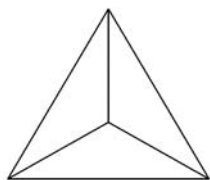
Solution. Were the polyhedron nonconvex, some rays from the focus might intersect it more than once, so the image would be ambiguous. Figure 8.5S.5 displays the diagrams. ♦

Exercise 24. Consider a regular polygon P with n edges. Describe the duals of the right prism with base P and the corresponding equilateral pyramid, dipyrmaid, and antiprism.

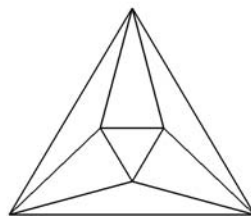
Solution. The prism and dipyrmaid are dual to each other; the pyramid is self-dual. The dual of the antiprism is constructed from $2n$ kites. ♦

Figure 8.5.11 Schlegel
diagram of a cube

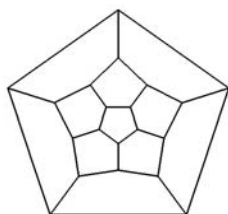




(a) Regular tetrahedron



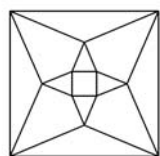
(b) Regular octahedron



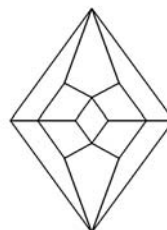
(c) Regular dodecahedron



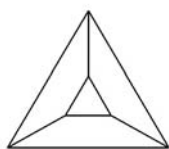
(d) Regular icosahedron



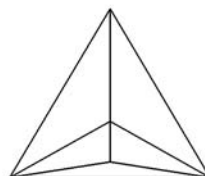
(e) Cuboctahedron



(f) Rhombic dodecahedron



(g) Equilateral triangular prism



(h) Equilateral triangular dipyramid

Figure 8.5S.5 (part 1 of 2)
Schlegel diagrams: exercise 8.5.23 solution

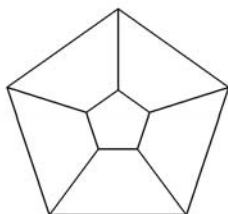
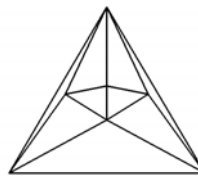
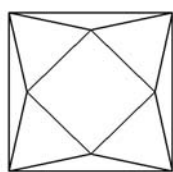
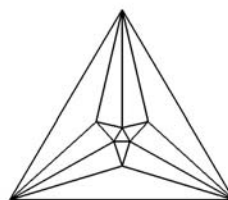
(i) Equilateral
pentagonal prism(j) Equilateral
pentagonal dipyrmaid(k) Square
antiprism(l) Deltahedrally stellated
triangular prism

Figure 8.5S.5 (part 2 of 2)
Schlegel diagrams: exercise 8.5.23 solution

Exercise 25, Part 1. Following the discussion under the section 8.4 heading **Regular polyhedra**, consider a regular octahedron $\mathcal{O}$ whose vertices are the points with coordinates ± 1 on the axes of a Cartesian coordinate system. Locate points on the edges of $\mathcal{O}$ that form the vertices of a regular icosahedron $\mathcal{I}$. Find the coordinates of the centroids of the faces of $\mathcal{I}$, and show that they form the vertices of a regular dodecahedron. *Suggestion:* You'll need the exact formula for the cosine of an interior angle of a regular pentagon—see exercise 4.9.1 and section 5.5.

Part 2. Compare this proof that a regular dodecahedron exists with Euclid's.¹⁴

Solution, Part 1. One face of the octahedron has centroid $C_0 = \langle \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \rangle$. On its edges locate points

$$A = \langle t, 1 - t, 0 \rangle \quad B = \langle 0, t, 1 - t \rangle \quad C = \langle 1 - t, 0, t \rangle,$$

¹⁴ Euclid [1908] 1956, book XIII, proposition 17.

where $t = -\frac{1}{2} + \frac{1}{2}\sqrt{5}$. These are the vertices of the icosahedral face $\triangle ABC$ with centroid C_0 . Neighboring icosahedral faces are $\triangle ZAB$ and $\triangle BCD$, where $Z = \langle 0, t, -1+t \rangle$ and $D = \langle -1+t, 0, t \rangle$, with centroids

$$C_1 = \left\langle \frac{t}{3}, \frac{1+t}{3}, 0 \right\rangle, \quad C_2 = \left\langle 0, \frac{t}{3}, \frac{1+t}{3} \right\rangle.$$

It's easy to check that $C_0 C_1 = C_0 C_2$. This situation is general enough to show that the centroids of the icosahedral faces are equidistant from each other. The point $E = \langle -t, 1-t, 0 \rangle$ forms a vertex of two more icosahedral faces $\triangle BDE$ and $\triangle BZE$. The centroid of the latter is

$$C_3 = \left\langle -\frac{t}{3}, \frac{1+t}{3}, 0 \right\rangle.$$

It's easy to check that C_0 to C_3 are coplanar. This situation is general enough to show that the centroids of the five icosahedral faces that share a common vertex are coplanar, hence they form the vertices of a dodecahedron, each of whose edges has length $C_3 C_1 = \frac{2}{3}t$. The cosines of the angle between $C_2 - C_0$ and $C_1 - C_0$ and that between $C_0 - C_1$ and $C_3 - C_1$ are

$$\frac{(C_2 - C_0) \cdot (C_1 - C_0)}{C_2 C_0 \cdot C_1 C_0} = \frac{t^2 - 3t + 1}{9}, \quad \frac{(C_3 - C_1) \cdot (C_0 - C_1)}{C_3 C_1 \cdot C_0 C_1} = \frac{2t(t-1)}{9}.$$

These are equal because $2t(t-1) - (t^2 - 3t + 1) = t^2 + t - 1 = 0$. In fact each of these cosines equals $\frac{1}{4} - \frac{1}{4}\sqrt{5} = \cos 108^\circ$, so C_0 to C_3 are vertices of a regular pentagon, and the dodecahedron is regular. ♦

Exercise 26. Define the notions *inradius* r and *circumradius* R of a regular polyhedron $\mathcal{P}$. The closer R/r is to 1, the closer $\mathcal{P}$ is to spherical. For each regular polyhedron derive a formula for this ratio. Do your results justify Plato's claim that the regular dodecahedron is most like a perfect sphere?

Solution. O will denote the centroid of $\mathcal{P}$; V a vertex; s its edge length; M an edge midpoint; and G , a , and b the centroid, inradius, and exradius of a face.

By exercise 5.11.7, for a regular tetrahedron, O lies $\frac{3}{4}$ of the way from V to the opposite G . Since $OV = R$ and $OG = r$, it follows that $R/r = 3$. For the cube, $R = \frac{1}{2}s\sqrt{3}$ and $r = \frac{1}{2}s$, so $R/r = \sqrt{3}$.

For the regular octahedron, $R = \frac{1}{2}s\sqrt{2}$ and $h = \frac{1}{2}s\sqrt{3}$. Right triangle $\triangle OGM$, where G and M lie on the same face, has legs r and $\frac{1}{3}s$ and hypotenuse $\frac{1}{2}s$. Pythagoras' theorem yields $r = \frac{1}{6}s\sqrt{6}$, hence $R/r = \sqrt{3}$.

Inspect a regular pentagon to determine $a = \frac{1}{2}s/\tan 36^\circ$ and $b = \frac{1}{2}s/\sin 36^\circ$. Use exact values for $\tan 36^\circ$ and $\sin 36^\circ$ calculated from exercise 4.9.1 or by computer algebra software. Locate figure 8.5S.6 in a regular

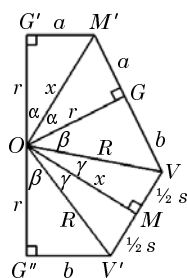


Figure 8.5S.6 Analyzing
a regular dodecahedron

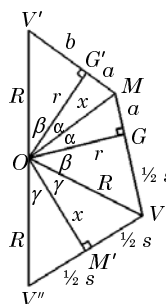


Figure 8.5S.7 Analyzing
a regular icosahedron

dodecahedron. First, notice that $2\alpha + 2\beta + 2\gamma = 180^\circ$ so that $\alpha + \beta + \gamma = 90^\circ$. Thus $OMVM'$ is a trapezoid and Pythagoras' theorem yields $x^2 + (x - \frac{1}{2}s)^2 = (a + b)^2$. Solve this equation for its positive root x , use $r^2 = x^2 - a^2$ to get r , $R^2 = r^2 + b^2$ to get R , and finally compute $R/r = \sqrt{15 - 6\sqrt{5}}$. (Use algebra software to manipulate the radicals.)

An equilateral triangular face of the regular icosahedron has altitude $h = \frac{1}{2}s\sqrt{3}$, so $a = \frac{1}{3}h = \frac{1}{6}s\sqrt{3}$ and $b = \frac{2}{3}h = \frac{1}{3}s\sqrt{3}$. Locate figure 8.5S.7 in a regular icosahedron. First, notice that $2\alpha + 2\beta + 2\gamma = 180^\circ$ so that $\alpha + \beta + \gamma = 90^\circ$. Thus $OMVM'$ is a trapezoid. The computations described in the previous paragraph lead to new values for x , r , and R , but the same ratio $R/r = \sqrt{15 - 6\sqrt{5}}$. In summary,

<i>Polyhedron</i>	<i>R/r</i>
Regular tetrahedron	3
Cube, regular octahedron	$\sqrt{3} \approx 1.7$
Regular dodecahedron, icosahedron	$\sqrt{15 - 6\sqrt{5}} \approx 1.3$

In this regard, the dodecahedron and icosahedron are *equally* near perfection! ♦

Exercise 27. For each regular polyhedron, compute the dihedral angle between adjacent faces. (You'll need this information for beveling should you plan to build finely joined models.)

Solution. For the tetrahedron, consider right triangle ΔVGM (using the notation of exercise 26) with G and M opposite V : the hypotenuse is the face altitude h , one leg is $\frac{1}{3}h$, and the angle θ between them measures the desired dihedral angle: $\cos \theta = \frac{1}{3}$, so $\theta = \arccos \frac{1}{3} \approx 70.53^\circ$.

For the cube, the dihedral angle is 90° . The discussion under the heading **Regular polyhedra** in section 8.4 shows that the dihedral angle of the octahedron is supplementary to that of the tetrahedron, hence it's approximately 109.47° . For the dodecahedron the dihedral angle θ is measured by $m\angle GM'G'$ in figure 8.5S.6. The computation described in the solution of exercise 26 yields

$$\theta = 2 \arccos a/x = 2 \arccos \sqrt{\frac{1}{2} - \frac{1}{10}\sqrt{5}} \approx 116.57^\circ.$$

For the icosahedron the dihedral angle θ is measured by $m\angle GMG'$ in figure 8.5S.7. The computation described in the solution of exercise 26 yields

$$\theta = 2 \arccos a/x = 2 \arccos \frac{\sqrt{15} - \sqrt{3}}{6} \approx 138.19^\circ. \blacklozenge$$

Exercise 28. The paragraph in section 8.4 that enumerates the symmetry group $\mathcal{T}_d$ of the regular tetrahedron only mentions eighteen isometries. What are the other six?

Solution. $\pm 90^\circ$ rotary reflections with axes joining midpoints of opposite edges. $\blacklozenge$

Exercise 29, Part 1. Each symmetry of a cube permutes the four lines through opposite vertices. The twenty-four rotational symmetries correspond bijectively with those permutations. Label the axes and choose angle parameters for the rotations conveniently, then describe such a bijection explicitly.

Part 2. Describe in detail the odd symmetries of a cube.

Solution, Part 2. There are 24 altogether. 9 are reflections, so 15 must be rotary reflections with nonzero angle parameters. $8 = 4 \times 2$ of these have mirrors halfway between opposite vertices. $6 = 3 \times 2$ have mirrors parallel to faces. The remaining one is the reflection across the cube's center. $\blacklozenge$

Exercise 30. Define the notion *interior* of a convex polyhedron. Show that the interior is a convex point set. Can you extend this notion to apply to any polyhedron?

Solution. Each face F of a convex polyhedron $\mathcal{P}$ determines a half space H_F , such that all vertices of $\mathcal{P}$ are on the face plane or in H_F . The interior of $\mathcal{P}$ is $\bigcap_F H_F$. It's convex because the half planes are. Approach the definition for polyhedra in general as in the solution to exercise 8.5.9. That isn't easy, and it requires reconciliation of the two definitions for the convex case. $\blacklozenge$

Stanislaus Ferdinand Victor SCHLEGEL was born in 1843 in Frankfurt am Main. He studied in Berlin during 1860–1863, and became a secondary-school teacher, serving from 1866 on in Breslau, Stettin, Mecklenburg, and Hagen. He earned the Ph.D. from the University at Leipzig in 1881, did some research on Grassmann's *Ausdehnungslehre* (a precursor of vector analysis) and other geometrical topics, and wrote many expository books and papers on mathematics for teachers and engineers. Schlegel turned to school administration in Hagen in 1893 and died there in 1905.

Exercise 31. Prove that for any face-connected simply connected polyhedron with v vertices, e edges, and f faces,

- | | |
|------------------------------------|---------------------|
| (a) $2e \geq 3v$ | (b) $2e \geq 3f$ |
| (c) $3f \geq 6 + e$ | (d) $3v \geq 6 + e$ |
| (e) $6v \geq 12 + 2e \geq 12 + 3v$ | (f) $2v \geq 4 + f$ |

Solution.

- (a) Count the two endpoints of each edge, getting $2e$. You'll count each vertex at least three times, because there are at least three edges at each vertex.
- (b) Count the two faces abutting each edge, getting $2e$. You'll count each face at least three times, because there are at least three edges of each face.
- (c) By (a) and then Euler's theorem, $\frac{2}{3}e + f \geq v + f = 2 + e$.
- (d) By (b) and then Euler's theorem, $\frac{2}{3}e + v \geq f + v = 2 + e$.
- (e) By (d) and then (a), $6v \geq 12 + 2e \geq 12 + 3v$.
- (f) By (e) and then (b), $2v \geq 4 + \frac{2}{3}e \geq 4 + f$. ♦

Exercise 32. Show that a face-connected simply connected polyhedron must have at least one face with fewer than six edges.

Solution. Otherwise, counting the two faces abutting each edge to get $2e$, you'll count each face at least six times, hence $2e \geq 6f$ in the notation of exercise 31. Part 3 of that exercise would then yield the absurdity $e \geq 3f \geq 6 + e$. ♦

Exercise 33, Part 1. Show that no face-connected simply connected polyhedron has exactly seven edges.

Part 2. For what n is there such a polyhedron with exactly n edges?

Solution, Part 1. If $e = 7$ in the notation of exercise 31, then by parts 2 and 3 of that exercise, $14 = 2e \geq 3f \geq 6 + e = 13$, which is impossible.

Part 2. For $n < 6$ there's no polyhedron. For even $n \geq 6$, let $m = \frac{1}{2}n$; a pyramid whose base has m edges has itself n edges. For $n = 7$

there's no such polyhedron. For odd $n > 7$, let $m = \frac{1}{2}(n - 3)$. A pyramid whose base has m edges has itself $n - 3$ edges. Based on one of its triangular faces erect a tetrahedron whose altitude is small enough that the resulting object $\mathcal{P}$ is still a face-connected simply connected polyhedron. $\mathcal{P}$ has n edges. ♦

Exercise 34. A European football is made of leather pieces in the shapes of regular pentagons and hexagons. They're sewn together so that each pentagon is surrounded by hexagons and each hexagon is surrounded alternately by pentagons and hexagons. From Euler's theorem, deduce how many pentagons and hexagons must be used. How many vertices and edges are there? How is the ball related to the regular polyhedra? What's the symmetry group of the ball? Draw its Schlegel diagram.

Solution. Let f_5 and f_6 denote the numbers of pentagons and hexagons, so that $f_5 + f_6 = f$ in the notation of exercise 31. Visit each edge, counting the two abutting faces; you'll get $2e = 5f_5 + 6f_6$. By the description, three edges meet at each vertex. Count the two endpoints of each edge, getting $2e = 3v$. Each vertex lies on exactly one pentagonal face, so $v = 5f_5$. Solve these four equations plus Euler's to get $f_5 = 12$, $f_6 = 20$, $f = 32$, $e = 90$, and $v = 60$. You can make the ball by truncating a regular icosahedron. It has an $\mathcal{I}_h$ symmetry group. Figure 8.5S.8 shows a Schlegel diagram for the ball. ♦

Exercise 35. Find and describe in detail all deltahedra with eight faces that aren't regular octahedra.

Solution. Use the notation of exercise 31 and let v_3 , v_4 , and v_5 denote the numbers of vertices incident with 3, 4, and 5 edges. Euler's theorem yields $v - e + 8 = 2$, hence $e = v + 6$. Visit each face, counting edges; you'll get $8 \cdot 3 = 2e$, hence $e = 12$ and $v = 6$. Visit each vertex, counting edges; you'll get $3v_3 + 4v_4 + 5v_5 = 2e = 24$. This equation has solutions $\langle v_3, v_4, v_5 \rangle = \langle 0, 6, 0 \rangle$, $\langle 1, 4, 1 \rangle$, $\langle 2, 2, 2 \rangle$, and $\langle 3, 0, 3 \rangle$. The first describes a regular octahedron. You can show that the second and last solutions

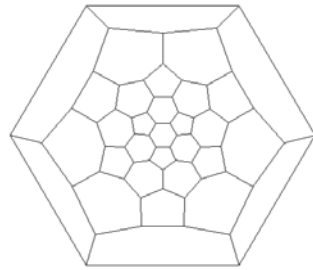


Figure 8.5S.8
Schlegel diagram for
a European football

can't describe any polyhedron—there's no way to connect the vertices like that. To form a polyhedron $\mathcal{T}$ with $v_3 = v_4 = v_5 = 2$, attach two regular tetrahedra $NOPQ$ and $PQRS$ along an edge $E = \overline{PQ}$, remove faces $\triangle OPQ$ and $\triangle PQR$, and add edge $\overline{OR}$ and faces $\triangle OPR$ and $\triangle OQR$. $\mathcal{T}$ looks like a tightly stretched tent; its symmetry group is $\mathcal{C}_{2v}$. ♦

A polyhedral vertex incident with exactly n edges is called *n-valent*.

Exercise 36. Show that there's no convex deltahedron with one 4-valent and ten 3-valent vertices.

Solution. If such a deltahedron $\mathcal{D}$ existed, the 4-valent vertex V would be adjacent to exactly four 5-valent vertices W, X, Y , and Z . Each of the faces $\triangle VWX$, $\triangle VXY$, $\triangle VYZ$, and $\triangle VZW$ would share its edge opposite V with another face different from those four. That would yield faces $\triangle WXR$, $\triangle XYS$, $\triangle YZT$, and $\triangle ZWU$. If $R = T$ or $S = U$, then $R = S = T = U$, else one of those edges would lie on more than two faces. But those equations would make W, X, Y , and Z 4-valent. Thus R, S, T , and U must be distinct, which yields four more faces $\triangle XRS$, $\triangle YST$, $\triangle ZTU$, and $\triangle WUR$. The nine vertices and twelve faces identified so far constitute part of the deltahedrally stellated square antiprism—what's lacking is the square pyramid with base $RSTU$. It would have one more vertex, which would be 4-valent. But $\mathcal{D}$ must have exactly two more vertices, each 5-valent. Each of these would be connected with the other and with R, S, T , and U . But that would make some edges shared by more than two faces. ♦

Exercise 37. Show that a convex deltahedron $\mathcal{D}$ with eight vertices, half 4-valent and half 5-valent, must be connected like the *Siamese dodecahedron* in figure 8.5.12. Show that there actually is a deltahedron with this structure.¹⁵

Solution. By Euler's theorem, $\mathcal{D}$ must have eighteen edges. No 4-valent vertex V can be surrounded by 5-valent vertices. For then, each adjacent pair of 5-valent vertices would have a common neighbor different from V , hence two opposite 5-valent vertices would have a common 4-valent neighbor. That requires more than eighteen edges. Thus, there must be two neighboring 4-valent vertices. Make two assemblies of three equilateral triangles as in figure 8.5S.8, with the center triangle joined flexibly to the outer triangles along edges $\overline{BC}$ and $\overline{CD}$. Now join the two assemblies together, flexibly, along edges $\overline{AB}$, $\overline{BD}$, and $\overline{DE}$. Flex the resulting

¹⁵ This deltahedron was studied first—at least in print—by Rausenberger (1915), then again by Freudenthal and van der Waerden (1947). Their publications addressed audiences of German and Dutch secondary school teachers. The latter authors named the polyhedron to suggest a resemblance to Siamese twins.

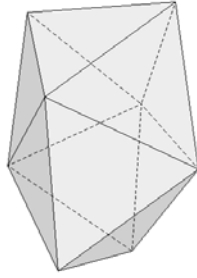


Figure 8.5.12
Siamese dodecahedron

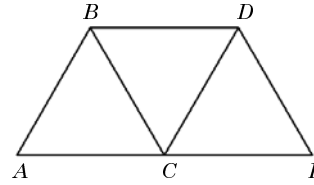


Figure 8.5S.9
“Tent” folded flat

structure so that it forms a sort of tent, which must be part of $\mathcal{D}$. Opposite C lies another vertex F joined by edges to A , B , D , and E . If A , C , E , or F were 4-valent in $\mathcal{D}$, it would “close” to form a regular octahedron, with too few faces. Therefore $\mathcal{D}$ must have two pairs of adjacent 4-valent vertices, each pair surrounded by 5-valent vertices. There must be two “tents” fitted together—the second upside down—along the zigzag edges joining A , C , E , and F .

Do two such “tents” actually fit together? The first one flexes continuously from the flat two-layer form in figure 8.5S.9, with AE twice edge length s and $CF = 0$, to a dipyrmaid with $AE = 0$ and $CF > 0$. A graph of AE against CF must cross the line $AE = CE$. (Details of the relationship between these quantities are irrelevant, as long it’s continuous.) Therefore you must be able to flex the tent so that segments $\overline{AE}$ and $\overline{CF}$ are congruent.¹⁶ Now let M and N be their midpoints, and consider right triangles $\triangle AMC$ and $\triangle ENC$. By Pythagoras’ theorem,

$$(AM)^2 + (MC)^2 = s^2 = (NC)^2 + (NE)^2.$$

From $AM = NC$ follows $MC = NE$, hence $\angle ACM \cong \angle NEC$, and $m\angle ACE = 2m\angle ACM = 2m\angle NEC = m\angle CEF$. That is, the “tents” fit together perfectly. ♦

Exercise 38. Identify the symmetry groups of the cuboctahedron, rhombic dodecahedron, the deltahedra in figures 8.4.25 and 8.4.26, and the Siamese dodecahedron.

Solution. Cuboctahedron, rhombic dodecahedron,
deltahedrally stellated square prism $\mathcal{O}_h$
Deltahedrally stellated triangular prism $\mathcal{D}_{3h}$
Siamese dodecahedron $\mathcal{D}_{2v}$. ♦

¹⁶ Different flexing yields the tent-shaped deltahedron of exercise 35.

PLATO was born about 427 B.C. in Athens, the son of noble, aristocratic parents. Through family connections he knew the philosopher Socrates, who was certainly the greatest influence on his development. The mathematician Theodorus of Cyrene was also in Athens during Plato's youth, and is reported to have been his teacher. After Socrates' execution in 399 B.C., Plato traveled for several years, then returned to Athens and founded there a school, the Academy, which he headed until his death about 347 B.C.

Under Plato's direction, the Academy became the center of studies in science, philosophy, and law; it was perhaps the first university. Its members included Theaetetus and Eudoxus, who made major discoveries in geometry essential to Euclid's later codification of that subject. A correspondent was the Pythagorean scientist and statesman Archytas of Tarentum, who was a major influence on Plato's own ideas.

Plato's writings, most in the form of dialogues, all survive, and have maintained a major influence on western philosophy ever since. They include inquiries into virtually all areas of the philosophy of his time. Plato was not himself a mathematician, so mathematics appears only occasionally in his works. The main example is in the dialogue *Timaeus*, in which Plato, through the words of the probably fictitious, vaguely Pythagorean title character, expounds his theory of the structure of matter based on properties of the regular polyhedra.

Timaeus was written during 368–348 B.C. (though its setting is some fifty years earlier). In 368 and 361 B.C., Plato visited cities in Sicily and in southern Italy, where Pythagoreans were influential, hoping to put into practice there his theory of the ideal political state. After meddling into local affairs, he was imprisoned, and apparently was extricated only through the efforts of his friend Archytas. Plato's political aspirations were never realized, but these escapades perhaps enhanced the Pythagorean influence on his late writings and on the mathematical work among the scholars at the Academy.

Plato's greatest student was Aristotle, who left the Academy to found his own school, the Lyceum. The Academy, however, continued in existence until A.D. 529.

Exercise 39. Construct a polyhedron with an $\mathcal{S}_{2n}$ symmetry group, in the style of figure 8.4.30. Show how to construct *convex* polyhedra with $\mathcal{C}_n$, $\mathcal{D}_n$, $\mathcal{S}_{2n}$, $\mathcal{C}_{nh}$, and $\mathcal{T}_h$ symmetry groups.

Solution. For a first polyhedron with $\mathcal{S}_{2n}$ symmetry group, see figure 8.5S.10. Yale suggests “shaved” prisms and antiprisms for the convex polyhedra.¹⁷ ♦

Exercise 40. Construct polyhedra with symmetry groups $\mathcal{C}_i$, $\mathcal{C}_{2h}$, $\mathcal{D}_2$, and $\mathcal{D}_{2v}$, as described in table 8.4.2.

¹⁷ [1968] 1988, section 3.2.

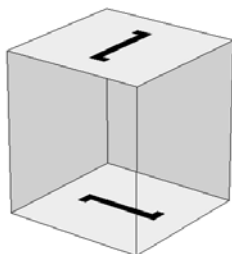


Figure 8.5S.10 Polyhedron
with $\mathcal{S}_{2n}$ symmetry group

Solution. For $\mathcal{C}_i$, a cube with L and Γ motifs centered on a pair of opposite vertical faces. For $\mathcal{C}_{2h}$, a cube with $\Rightarrow$ motifs centered on a pair of opposite vertical faces. For $\mathcal{D}_2$, a cube with / motifs centered on a pair of opposite vertical faces. For $\mathcal{D}_{2v}$, a cube with V motifs centered on a pair of opposite vertical faces and Λ motifs centered on the other two. ♦

Exercise 41. Use this book's references, your library, museums, commercial shops, and buildings in your neighborhood to find functional and ornamental examples of polyhedra with as many different symmetry groups as possible. Expand the selection in exercises 1 and 2.

Solution. Not available. ♦