

1. *Assignment*
 - a. Are there further questions about the social organization of mathematics?
 - b. Continue reading Gillman 1987 and prepare to discuss it.
 - c. Various students are about ready to start presenting outlines. Let's schedule them.
2. *In-class proof.* Here's the completed proof.

Proposition. Suppose A is a subset that spans a vector space V , and B is a basis of V . Then $\#B \leq (\#A)\omega$.

*Proof.*¹ Let $\mathcal{E}$ be the family of all nonempty finite subsets of B . By the finiteness principle, each $x \in A$ depends on a finite subset $S \in \mathcal{E}$ (because B is a basis). That is, x belongs to the span $\bar{S}$ of S , so that $\{S \in \mathcal{E} : x \in \bar{S}\}$ is nonempty. By the axiom of choice,² there is a choice function $E \in \prod_{x \in A} \{S \in \mathcal{E} : x \in \bar{S}\}$. Thus $x \in \bar{E_x}$ for each $x \in A$. Since A spans V and $A \subseteq \bigcup_{x \in A} \bar{E_x}$, it is also true that $\bigcup_{x \in A} E_x$ spans V . Moreover, $\bigcup_{x \in A} E_x = B$ since $\bigcup_{x \in A} E_x \subseteq B$ and B is a basis of V . The cardinal $\#B$ of $\bigcup_{x \in A} E_x$ is $\leq$ the cardinal of the union of disjoint copies of the sets E_x for $x \in A$, which is the sum $\sum_{x \in A} (\#E_x)$ of the cardinals $\#E_x$. Since E_x is finite, $\#E_x \leq \omega$ for each $x \in A$, so $\sum_{x \in A} (\#E_x) \leq \sum_{x \in A} \omega$ by standard cardinal arithmetic.³ Finally, $\sum_{x \in A} \omega = (\#A)\omega$, the cardinal product.

Corollary. If A and B are bases of V , then $\#A = \#B$.

Proof. If A or B is finite, this follows from a result proved elsewhere. If both are infinite, then $(\#A)\omega = \#A$ and $\#B\omega = \#B$ by standard cardinal arithmetic, and thus the previous theorem implies $\#B \leq (\#A)\omega = \#A \leq (\#B)\omega = \#B$.

3. This example proof should have demonstrated some techniques that are applied constantly.
 - a. How easy it was to convert to flowing English once the proof (or any discussion, for that matter) has been cast as a very detailed outline! For that reason, refine the *outline*, not the prose! Wait until the last possible moment to render your paper in flowing English.
 - b. Mathematical variables are almost *always* italicized.
 - c. Parentheses are almost *never* italicized.
 - d. Inline mathematical formulas are always set off with extra white space fore and aft.

¹ The proof follows Zariski and Samuel 1958–1960, volume 1, 99.

² Zariski and Samuel presented an involved construction of E without using the axiom of choice.

³ Verifying this inequality in detail requires use of the axiom of choice. For cardinal arithmetic used here, consult Suppes 1960.

4. *Gillman 1987, chapter 3.*
 - a. Proving P “by contradiction”—that is, by assuming $\neg P$ and deriving a result that contradicts something established earlier—is often awkward and easily misunderstood.
 - b. In very many cases, P actually has the form $Q \Rightarrow R$, and you can recast the awkward proof into a graceful, more easily understood proof of the contrapositive $\neg R \Rightarrow \neg Q$, which is equivalent to P .