

1. *Assignment*
 - a. Have we neglected anything for the in-class outline?
 - b. Are there further questions about the social organization of mathematics?
 - c. Begin reading Gillman 1987. I posted a portion of it temporarily on the course website. Ms. Morgan and Mr. Kifer have agreed to lead our discussion of the first sections of this manual.
2. A question arose in class about use of the passive voice. I just noticed a good example the account of a Math 880 discussion last year. It concerned an editing technique: surgery on individual sentences or pairs of sentences to shorten them without changing the semantics. Applied to something written with less care, this can often cut the length significantly. This sort of editing is often extremely beneficial in reducing clutter in abstracts and presentations, and in increasing effectiveness in general writing. The example was from an abstract in our elevator lobby intended to attract an audience to a talk by a candidate for our then-open chairmanship. (The candidate was not successful.)
 - a. First sentence: “Recent advances in computational modeling and imaging technology are making it possible that computational models can be built based on patient-specific image data and accurate predictions can be made for realistic clinical applications.” (33 words, 205 characters)
 - b. Edited version: “Recent technological advances let us base computational models on patient-specific image data to make accurate predictions for clinical applications.” (19 words, 130 characters)
 - c. This is a 42% or 37% reduction, depending on what you count.
 - d. Repeated phrases and use of passive voice in a passage are indicators that this sort of editing can help. Also suggestive are indirections such as “are making it possible that ... can be built,” which I generally describe uncharitably as ways to avoid responsibility for what one says. Those eight words here are replaced by three: “let us build...”
 - e. Passive voice is overused as a way of avoiding writer’s responsibility. It’s a mark of a bureaucrat-in-training.
3. I digressed to explain how the ability to do linear algebra without requiring commutativity of the scalars benefits the study of foundations of geometry.
 - a. A common strategy is to develop a synthetic axiom system (about points, lines, planes, etc.) sufficiently to introduce the scalars as points on a given line, define their addition and multiplication, prove the major properties of those operations, assign coordinates to points, and show that lines and planes are graphs of linear equations of some sort. For the latter proofs, it’s nice to use tools of linear algebra. If those were available only for commutative scalars, we would have to prove commutativity before dealing with linear equations. Although commutativity of scalars follows easily from the famous Pappus-Pascal theorem about collinearity of the intersections of opposite edge lines

of a hexagon inscribed in a degenerate conic, that theorem is very hard to prove from synthetic axioms. That's why it doesn't appear in school geometry texts. If we build the linear algebra tools without assuming commutativity, we can show that most of analytic geometry is independent of the apparatus necessary to prove the Pappus-Pascal theorem.

4. *In-class outline.* We discussed inclusion of examples in the outline.
 - a. Some organization is required, since some examples depend on previous ones, and some could be included in more than one place.
 - b. I'll put some very brief placeholders for examples into the outline.
 - c. The current state of the outline is included at the end of this document.
5. *Materials on mathematical writing.* Last year, students and I reviewed some such guides, made some comments, and decided to discuss part of one of them in detail, together in class. Here I'll mention those we considered. I've posted Gillman 1987 online temporarily, so that you can read ahead and be prepared to start a discussion of it during the next meeting.
 - a. *Chicago 1993* is the bible. Both Wiley and Birkhäuser simply told me to adhere to its rules and suggestions. You should become acquainted with it.
 - b. *Finkel et al. 2000* has some interesting stuff, intended for applied mathematics PhD students. Some attention to abstracts. It refers to the research paper Stewart 2000 and its abstract, Trefethen 2000.
 - c. *Fowler 1958* is a wonderful source of tips on writing English. It's divided into separate articles on topics such as "than". For that reason, you'll find it perplexing: the question you're asking is hard to find. Solution: use it for bedtime reading. It's often hilarious, and that makes its tips memorable.
 - d. *Gillman 1987* seems hard to obtain, but is concise, memorable, and appeared the most useful. The author, a noted researcher in functional analysis, was editor of the *American Mathematical Monthly*. We'll discuss about 2/3 of this in class. It's on my website temporarily.
 - e. *Higham 1993* is more comprehensive than Gillman 1987, but did not seem to generate as much interest.
 - f. *Krantz 2005* is mostly unsuited for class discussion, but its chapter 5 does have some very interesting stuff on a wide variety of topics. Krantz will give a colloquium talk here this spring.
 - g. *Steenrod et al. 1973* includes an article on writing by Halmos that is memorable, but not really close to my taste.
 - h. *Strunk 2000* is a modern edition of a classic style guide often referred to as *Strunk and White*, that served as a model for many later style guides. I suppose I've read and used it in the past, but don't seem to have a copy now.
6. Here's the current state of the outline.
 - a. *Introduction*
 - i. This paper presents the main features of the theory of linear dependence.
 - (1) This theory includes several of the major concepts and theorems of linear algebra,

- (2) commonly introduced in first courses in linear algebra.
- ii. But its context is considerably more general than that,
 - (1) to permit a much broader scope of application.
- iii. The theory is presented in the context of a vector space V over a division ring K of scalars.
 - (1) Footnote: an alternative term for *division ring* is *skew field*.
 - (2) Postulates for a division ring.
 - (3) Examples: real numbers, complex numbers, rational numbers. (Others later.)
 - (4) Postulates for a vector space V over a division ring K , whose elements are called *scalars*.
 - (5) Examples: real and complex n -space. (Others later.)
 - (6) Definition of a subspace of V .
 - (7) Examples: $\{0\}$, V , lines and planes given by parametric equations in any n -space, hyperplanes and hyperlines given by linear equations in any n -space.
- iv. The presentation follows that of Van der Waerden [1937] 1953.
 - (1) In this framework,
 - (a) dependence of a vector on a set of vectors is defined first,
 - (b) and few basic theorems ($\exists$ A BETTER TERM ?) are proved.
 - (c) All subsequent definitions and theorems are based solely on these.
 - (2) This permits all these definitions and theorems to be carried intact into the development of any theory that includes an analogous concept of dependence that satisfies those few basic theorems.
 - (3) Moreover, by eliminating distracting use of methods specific to vector spaces, the framework encourages presentation of the simplest proofs.
- v. The present context for this theory is in fact considerably more general than that of a first linear-algebra course.
 - (1) Rather than requiring scalars to be real or complex numbers,
 - (2) it permits use of scalars in some finite division rings, such as $\mathbb{Z}_p$ for prime p . The two-element ring $\mathbb{Z}_2 = \{0, 1\}$ is often used in the theory of computation.
 - (3) And it even permits the scalars to lie in a noncommutative division ring, such as the quaternions. This permits use of linear algebra at an earlier stage in the study of the foundations of geometry than would be possible otherwise, since commutativity of scalars is hard to derive from geometric axiom systems.
 - (4) Rather than requiring the vector space to be finite-dimensional, the proofs presented here apply even to infinite-dimensional spaces. This permits use of linear algebra in studying the spaces of polynomials, formal power series, and continuous functions, for example.

The examples could also include spaces of sequences of any finite length, of all sequences, of all convergent sequences, etc.

- vi. MAYBE AN EXPLANATION IS NEEDED HERE THAT THIS PAPER WILL NOT ATTEMPT TO COVER THE THEORY OF MODULES OVER RINGS, WHICH SATISFY ALL THE POSTULATES FOR VECTORS AND SCALARS EXCEPT ...
- vii. MAYBE A PARAGRAPH IS NEEDED HERE TO ENUMERATE THE SUBSEQUENT SECTIONS OF THE PAPER.
- b. *Van der Waerden's framework*
 - i. Explain about the framework, in Van der Waerden [1937] 1953, section 33.
 - ii. Define *dependence* of a vector v on a set S of vectors.
 - iii. Example from $\mathbb{R}^3$.
 - iv. Note that each $v \in S$ depends on S .
 - v. Show that v depends on S if and only if it depends on a finite subset of S .
 - vi. Show that if each member of a set R of vectors depends on S , then each vector that depends on R also depends on S .
 - vii. *Exchange Lemma*. If a vector v is not dependent on a set S of vectors, but is dependent on $S \cup \{u\}$ for some vector u , then u is dependent on $S \cup \{v\}$.
 - (1) *Proof*. v would be the sum of a linear combination of vectors in X and tu for some scalar t . But t must be nonzero, so has an inverse, and u would thus be a linear combination of vectors in $S \cup \{v\}$.
 - (2) Example from $\mathbb{R}^3$.
 - viii. SHOULD WE REPEAT HERE THAT WE'LL BASE ALL FURTHER RESULTS ONLY ON THE NOTION OF DEPENDENCE AND THE PROPERTIES JUST PROVED?
- c. *Bases*
 - i. *Span and independence*
 - (1) Define *span* $\bar{S}$ of a set S of vectors.
 - (2) State and prove some properties of the mapping $S \rightarrow \bar{S}$. For example, that it's a closure operator, and that spans are subspaces.
 - (3) Define a set S of vectors to be *independent* if no $v \in S$ depends on $S - \{v\}$.
 - (4) Example from $\mathbb{R}^3$ that's not a basis.
 - ii. *Theorem*. The following properties of a set B of vectors are equivalent:
 - (1) S is independent and span the whole vector space V .
 - (2) B is a maximal independent subset of V .
 - (3) B is a minimal spanning subset of V .
 - iii. *Definition*. A subset with these properties is called a *basis* of V .
 - iv. Examples from spaces of various dimensions, including that of finite sequences of any length.
 - v. *Theorem*. Every independent set S of vectors is contained in a basis.

- (1) *Proof.* Show that the family $\mathcal{F}$ of all independent sets containing S satisfies the hypotheses of Zorn's lemma. Zorn's lemma then implies that $\mathcal{F}$ has a maximal member.
 - vi. *Corollary.* Every vector space has a basis.
 - (1) *Proof.* The empty family is independent.
 - vii. Note that it's not clear what a basis of the space of *all* sequences might be.
 - viii. *Theorem.* Every set S of vectors contains a basis of the span of S .
 - (1) *Proof.* This is a little more complicated, and uses the exchange lemma.
 - ix. *Exchange Theorem.* If S, S' are bases then for every $b \in B$ there exists $b' \in B'$ such that $(B - \{b\}) \cup \{b'\}$ is a basis.
 - (1) *Proof.* This, too, uses the exchange lemma.
 - x. Examples from $\mathbb{R}^3$ and from some higher dimension.
 - xi. *Corollary.* All bases are equinumerous.
 - (1) The proof in my notes uses the axiom of choice to well-order the bases, then uses recursion on the order relations, with the exchange theorem, to construct a bijection from one base to the other.
 - (2) I SUSPECT THAT THIS CAN BE SHORTCUT TO USE ZORN'S LEMMA, ANOTHER CONSEQUENCE OF THE AXIOM OF CHOICE, INSTEAD OF WELL-ORDERING AND RECURSION. THAT WOULD BE A GOOD TOPIC FOR THE WRITER TO INVESTIGATE.
- d. *Dimension*
- i. *Definition.* The dimension of a vector space is the common cardinality of its bases.
 - ii. Examples: K^n has dimension n . The space of finite sequences of any length has dimension ω . We might give the dimension 2^ω of the space of all sequences of real numbers, but I don't remember now the proof of that. This would be a good question for the writer to investigate.
 - iii. *Corollary.* $\dim(X \cap Y) + \dim(X \vee Y) \leq \dim X + \dim Y$ for any subspaces X, Y .
 - (1) *Proof.* This comes from repeated use of the theorem about extending independent subsets to bases, then counting.
 - (2) My notes are set in a context more general than vector spaces. For vector spaces, equality may hold here. I'm not sure.
 - (3) Examples?
 - (4) My notes give some conditions under which equality does hold. I think they are met in the finite-dimensional case. Investigating the infinite dimensional case would be a good project for the writer.
 - iv. *Rank & nullity theorem*
 - (1) My notes provide a general proof that when X is a subspace of a space Y , then $\dim X + \dim(Y/X) = \dim Y$, where Y/X is, at least, closely related to the usual quotient space.

- (2) I think that leads in the finite dimensional case to the theorem, if f is a linear map from one space V to another, then $\dim V = \dim \text{Range } f + \dim \text{Kernel of } f$.
- (3) Examples?
- (4) I DON'T KNOW WHETHER PRECISELY THAT THEOREM HOLDS IN INFINITE DIMENSIONAL LINEAR ALGEBRA. INVESTIGATING THAT WOULD BE A GOOD PROJECT FOR THE WRITER.
- e. *History*. CONSIDER WHETHER AND WHERE TO INSERT HISTORY IN THIS OUTLINE.
- f. *Further applications*. WE MAY WANT TO MENTION SOME APPLICATIONS OF THESE THEOREMS OUTSIDE LINEAR ALGEBRA.
- g. *Conclusion*. THERE SHOULD BE A CONCLUDING PARAGRAPH.
- h. *References*

Baer, Reinhold. 1952. *Linear Algebra and Projective Geometry*. Pure and applied mathematics, 2. New York: Academic Press.

Van der Waerden, B. L. [1937] 1953. *Modern Algebra*, in part a development from lectures by E. Artin and E. Noether, volume 1. Translated from the second German edition by Fred Blum, with revisions and additions by the author. Revised English edition. New York: Frederick Ungar Publishing Co. Originally published as *Moderne Algebra* in Berlin, 1931, by Julius Springer.