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1. *Assignment*
 - a. Have we neglected anything for the in-class outline?
 - b. Continue formulating questions about the social organization of mathematics.
 2. *A typesetting problem*
 - a. In setting the in-class outline I used the conventional notation $\bar{S}$ for the span of S . This was hard.
 - b. That letter $\bar{S}$ is included among those provided by *WordPerfect*, but only with the S in a different typeface from what I'm using here. It jars.
 - c. I tried *MathType*, which lets me use my typeface, but its overline is too high and hits the text above if badly placed: $\bar{S}$.
 - d. Finally, I solved the problem just with *WordPerfect* by typing S then inserting a vertical move 0.1 inch up, a horizontal move 0.08 inch left, typing an en dash, moving 0.1 inch down, and typing the period: $\bar{S}$.
 - e. This solution generalizes to longer overlines. You can use an em dash, or more than one in succession.
 - f. I believe a similar solution is available in *Word*, but is much more awkward. L^AT_EX fans assure me that it takes such problems in stride: if it doesn't get the overline quite right, you can reprogram it! But of course, that's exactly what I did with *WordPerfect*, in a probably easier way.
 3. *In-class outline*
 - a. We reformulated the outline of the paper's introduction.
 - b. It probably wasn't appropriate to do that before, because it was not clear what would need to be introduced.
 - c. There are still some questions left about the scope of the paper. Those may lead to some inclusions that may then require some additional sentences in the introduction.
 - d. I massaged that discussion, edited the outline accordingly, have posted the new version, and include it below as a snapshot of progress to date.
 - e. It's really filling out, isn't it.
 - f. There are only a few questions remaining that we should still consider:
 - i. Some are minor, involving only exposition tactics.
 - ii. Some would require some investigation and are left to our imaginary writer.
 - iii. Others we should discuss in class.
 - (1) Where should we insert examples into this outline?
 - (2) Should we insert history into this outline? If so, where?
 - (3) Should we mention any applications of these theorems outside linear algebra?
 - iv. What should a concluding section look like?
 - v. What's the title?
 4. Here's the current state of the outline.

a. *Introduction*

- i. This paper presents the main features of the theory of linear dependence.
 - (1) This theory includes several of the major concepts and theorems of linear algebra,
 - (2) commonly introduced in first courses in linear algebra.
- ii. But its context is considerably more general than that,
 - (1) to permit a much broader scope of application.
- iii. The theory is presented in the context of a vector space V over a division ring K of scalars.
 - (1) Footnote: an alternative term for *division ring* is *skew field*.
 - (2) Postulates for a division ring.
 - (3) Postulates for a vector space V over a division ring, whose elements are called *scalars*.
 - (4) Definition of a subspace of V .
- iv. The presentation follows that of Van der Waerden [1937] 1953.
 - (1) In this framework,
 - (a) dependence of a vector on a set of vectors is defined first,
 - (b) and few basic theorems (¿ $\exists$ A BETTER TERM ?) are proved.
 - (c) All subsequent definitions and theorems are based solely on these.
 - (2) This permits all these definitions and theorems to be carried intact into the development of any theory that includes an analogous concept of dependence that satisfies those few basic theorems.
 - (3) Moreover, by eliminating distracting use of methods specific to vector spaces, the framework encourages presentation of the simplest proofs.
- v. The present context for this theory is in fact considerably more general than that of a first linear-algebra course.
 - (1) Rather than requiring scalars to be real or complex numbers,
 - (2) it permits use of scalars in some finite rings, even from the ring $\{0, 1\}$ often used in the theory of computation.
 - (3) And it even permits the scalars to lie in a noncommutative division ring, such as the quaternions. This permits use of linear algebra at an earlier stage in the study of the foundations of geometry than would be possible otherwise.
 - (4) Rather than requiring the vector space to be finite-dimensional, the proofs presented here apply even to infinite-dimensional spaces. This permits use of linear algebra in studying the spaces of polynomials, formal power series, and continuous functions, for example.
- vi. MAYBE AN EXPLANATION IS NEEDED HERE THAT THIS PAPER WILL WILL NOT ATTEMPT TO COVER THE THEORY OF MODULES OVER RINGS, WHICH SATISFY ALL THE POSTULATES FOR VECTORS AND SCALARS EXCEPT ...

- vii. MAYBE A PARAGRAPH IS NEEDED HERE TO ENUMERATE THE SUBSEQUENT SECTIONS OF THE PAPER.
- b. *Van der Waerden's framework*
 - i. Explain about the framework, in Van der Waerden [1937] 1953, section 33.
 - ii. Define *dependence* of a vector v on a set S of vectors.
 - iii. Note that each $v \in S$ depends on S .
 - iv. Show that v depends on S if and only if it depends on a finite subset of S .
 - v. Show that if each member of a set R of vectors depends on S , then each vector that depends on R also depends on S .
 - vi. *Exchange Lemma*. If a vector v is not dependent on a set S of vectors, but is dependent on $S \cup \{u\}$ for some vector u , then u is dependent on $S \cup \{v\}$.
 - (1) *Proof*. v would be the sum of a linear combination of vectors in X and tu for some scalar t . But t must be nonzero, so has an inverse, and u would thus be a linear combination of vectors in $S \cup \{v\}$.
 - vii. SHOULD WE REPEAT HERE THAT WE'LL BASE ALL FURTHER RESULTS ONLY ON THE NOTION OF DEPENDENCE AND THE PROPERTIES JUST PROVED?
- c. *Bases*
 - i. *Span and independence*
 - (1) Define *span* $\bar{S}$ of a set S of vectors.
 - (2) State and prove some properties of the mapping $S \rightarrow \bar{S}$.
 - (3) Define a set S of vectors to be *independent* if no $v \in S$ depends on $S - \{v\}$.
 - ii. *Theorem*. The following properties of a set B of vectors are equivalent:
 - (1) S is independent and span the whole vector space V .
 - (2) B is a maximal independent subset of V .
 - (3) B is a minimal spanning subset of V .
 - iii. *Definition*. A subset with these properties is called a *basis* of V .
 - iv. *Theorem*. Every independent set S of vectors is contained in a basis.
 - (1) *Proof*. Show that the family $\mathcal{F}$ of all independent sets containing S satisfies the hypotheses of Zorn's lemma. Zorn's lemma then implies that $\mathcal{F}$ has a maximal member.
 - v. *Corollary*. Every vector space has a basis.
 - (1) *Proof*. The empty family is independent.
 - vi. *Theorem*. Every set S of vectors contains a basis of the span of S .
 - (1) *Proof*. This is a little more complicated, and uses the exchange lemma.
 - vii. *Exchange Theorem*. If S, S' are bases then for every $b \in B$ there exists $b' \in B'$ such that $(B - \{b\}) \cup \{b'\}$ is a basis.
 - (1) *Proof*. This, too, uses the exchange lemma.
 - viii. *Corollary*. All bases are equinumerous.

- (1) The proof in my notes uses the axiom of choice to well-order the bases, then uses recursion on the order relations, with the exchange theorem, to construct a bijection from one base to the other.
 - (2) I SUSPECT THAT THIS CAN BE SHORTCUT TO USE ZORN'S LEMMA, ANOTHER CONSEQUENCE OF THE AXIOM OF CHOICE, INSTEAD OF WELL-ORDERING AND RECURSION. THAT WOULD BE A GOOD TOPIC FOR THE WRITER TO INVESTIGATE.
- d. *Dimension*
- i. *Definition.* The dimension of a vector space is the common cardinality of its bases.
 - ii. *Corollary.* $\dim(X \cap Y) + \dim(X \vee Y) \leq \dim X + \dim Y$ for any subspaces X, Y .
 - (1) *Proof.* This comes from repeated use of the theorem about extending independent subsets to bases, then counting.
 - (2) My notes are set in a context more general than vector spaces. For vector spaces, equality may hold here. I'm not sure.
 - (3) My notes give some conditions under which equality does hold. I think they are met in the finite-dimensional case. Investigating the infinite dimensional case would be a good project for the writer.
 - iii. *Rank & nullity theorem*
 - (1) My notes provide a general proof that when X is a subspace of a space Y , then $\dim X + \dim(Y/X) = \dim Y$, where Y/X is, at least, closely related to the usual quotient space.
 - (2) I think that leads in the finite dimensional case to the theorem, if f is a linear map from one space V to another, then $\dim V = \dim \text{Range } f + \dim \text{Kernel of } f$.
 - (3) I DON'T KNOW WHETHER PRECISELY THAT THEOREM HOLDS IN INFINITE DIMENSIONAL LINEAR ALGEBRA. INVESTIGATING THAT WOULD BE A GOOD PROJECT FOR THE WRITER.
- e. *Examples.* CONSIDER WHERE TO INSERT EXAMPLES IN THIS OUTLINE.
- f. *History.* CONSIDER WHETHER AND WHERE TO INSERT HISTORY IN THIS OUTLINE.
- g. *Further applications.* WE MAY WANT TO MENTION SOME APPLICATIONS OF THESE THEOREMS OUTSIDE LINEAR ALGEBRA.
- h. *Conclusion.* THERE SHOULD BE A CONCLUDING PARAGRAPH.
- i. *References*

Baer, Reinhold. 1952. *Linear Algebra and Projective Geometry*. Pure and applied mathematics, 2. New York: Academic Press.

Van der Waerden, B. L. [1937] 1953. *Modern Algebra*, in part a development from lectures by E. Artin and E. Noether, volume 1. Translated from the second German edition by Fred Blum, with revisions and additions by the author. Revised English edition. New York: Frederick Ungar Publishing Co. Originally published as *Moderne Algebra* in Berlin, 1931, by Julius Springer.