

1. *Assignment*
 - a. Continue formulating questions about the paper we're outlining in class.
 - i. Where should examples go?
 - ii. What about history?
 - iii. What general cleanup is appropriate?
 - b. Continue formulating questions about the social organization of mathematics.
2. *In-class outline*
 - a. I reviewed the informal paper I wrote on this subject in 1968 as a graduate student and found that the basic properties of dependence of vectors were stated in Van der Waerden [1931] 1953, volume 1, 99 ff. That was in fact the text in the undergraduate course I took in 1959 that was analogous to Math 335. Its first edition is considered the first comprehensive modern-algebra text. Here are the properties:
 - i. All points in a finite set F of vectors depend on F .
 - ii. If E, F are finite sets of vectors and each member of E depends on F , then each vector dependent on E depends on F .
 - iii. If E is a finite set of vectors, q is a vector, and p is a vector dependent on $F \cup \{q\}$ but not on F , then q depends on $F \cup \{p\}$.
 - iv. A vector depends on a set X of vectors if and only if it depends on some finite subset of X .
 - b. Van der Waerden didn't actually state property (iv), because he only considered dependence on finite sets. But many authors after him noticed that his restriction wasn't necessary. Van der Waerden proceeded to derive the main theorems of linear algebra from those properties, and applied them to the study of algebraic dependence in fields.
 - c. We revised the outline, incorporating this framework just after the introduction. I massaged it a little further, to clean up some minor language and organization flaws.
 - d. This will cause us to rethink the introduction, later.
 - e. We also organized the long list of remaining definitions and results into two groups, on bases and on dimension.
 - f. Finally, we inserted a placeholder for a reference to Van der Waerden [1937] 1953, and I have constructed that reference.
 - g. The current state of the outline is posted online and repeated at the end of the current document.
3. *Compound surnames*
 - a. I learned recently that Dutch surnames beginning with *Van* should be alphabetized under *Van*. So I've had to change my practice.
 - b. This is confusing, because German names such as John von Neumann are alphabetized as *Neumann*, *John von*. And then Italian surnames such as *De Paolis* are alphabetized under *De*.

- c. Such details make a difference, because in a large bibliography, *De Paolis* and *Paolis* might be many pages apart!
 - d. Punctuation marks in surnames such as *Whiting-O'Keefe* are equally perplexing. Moreover, there are blanks in the middle of *Van der Waerden*. My current practice is to treat all punctuation marks and blanks as though they weren't there. That causes *Van der Bilt* and *Vanderbilt* to be treated identically, which is what you want.
 - e. Occasionally names occur that don't fit this scheme, or exceptions must be made to accord with widespread public use. The goal of the rules in a manual such as Chicago 1993 is to minimize the number of *ad hoc* exceptions.
4. *Van der Waerden [1937] 1953*
- a. I include both dates because I used the 1953 book, while the 1937 version may be historically significant. We might want to discuss when this general framework arose. Actually it might have been in the original 1931 addition, in which case I'd use the dates [1931] 1953. I'd also do that if we wished to discuss the influence of this book on the development of modern-algebra courses. It might also be better to cite an English edition more recent than 1953, with an ISBN number.
 - b. I included the language about Artin and Noether, and about the translation, because it's historically significant. I used exactly the language on the title page, rather than spend my time editing it.
 - c. Should first names be used? For me that has been a perplexing question.
 - i. Sometimes they're required to distinguish people. For example, there are two famous mathematicians named Noether, and two named Artin!
 - ii. Sometimes they're hard to find: *Emil* Artin and *Emmy* Noether are easy. It's rather harder to find *Bartel Leendert* van der Waerden!
 - iii. Sometimes that search leads to distracting results, such as *Julius Wilhelm Richard* Dedekind, whom everyone knows merely as *Richard*.
 - d. I included the original publication information as a favor to readers who might want to find out how something was originally phrased.
 - e. I omitted some interesting lore from this reference, because it's probably irrelevant. The title and copyright pages identify the book as published and sold in the public interest as property confiscated from Springer during World War II!
5. Here's the current state of the in-class outline.
- a. *Introduction*. THIS NEEDS RETHINKING.
 - i. ??Should we explain the basics?
 - (1) vector space
 - (2) independent subset
 - (3) spanning subset
 - ii. The idea of the following arrangement is to base all the initial results on the ideas in the introduction, and on the following
 - b. *Van der Waerden's framework*

- i. Explain about the framework, in Van der Waerden [1937] 1953, section 33.
 - ii. Define *dependence* of a vector v on a set S of vectors.
 - iii. Note that each $v \in S$ depends on S .
 - iv. Show that v depends on S if and only if it depends on a finite subset of S .
 - v. Show that if each member of a set R of vectors depends on S , then each vector that depends on R also depends on S .
 - vi. *Exchange Lemma*. If a vector v is not dependent on a set S of vectors, but is dependent on $S \cup \{u\}$ for some vector u , then u is dependent on $S \cup \{v\}$.
 - (1) *Proof*. v would be the sum of a linear combination of vectors in X and tu for some scalar t . But t must be nonzero, so has an inverse, and u would thus be a linear combination of vectors in $S \cup \{v\}$.
 - vii. SHOULD THERE BE A RECAPITULATION HERE, THAT WE'LL BASE ALL FURTHER RESULTS ONLY ON THE NOTION OF DEPENDENCE AND THE PROPERTIES JUST PROVED?
- c. *Bases*
- i. *Span and independence*
 - (1) Define *span* $\bar{S}$ of a set S of vectors.
 - (2) State and prove some properties of the mapping $S \rightarrow \bar{S}$.
 - (3) Define a set S of vectors to be *independent* if no $v \in S$ depends on $S - \{v\}$.
 - ii. *Theorem*. The following properties of a set B of vectors are equivalent:
 - (1) S is independent and span the whole vector space V .
 - (2) B is a maximal independent subset of V .
 - (3) B is a minimal spanning subset of V .
 - iii. *Definition*. A subset with these properties is called a *basis* of V .
 - iv. *Theorem*. Every independent set S of vectors is contained in a basis.
 - (1) *Proof*. Show that the family $\mathcal{F}$ of all independent sets containing S satisfies the hypotheses of Zorn's lemma. Zorn's lemma then implies that $\mathcal{F}$ has a maximal member.
 - v. *Corollary*. Every vector space has a basis.
 - (1) *Proof*. The empty family is independent.
 - vi. *Theorem*. Every set S of vectors contains a basis of the span of S .
 - (1) *Proof*. This is a little more complicated, and uses the exchange lemma.
 - vii. *Exchange Theorem*. If S, S' are bases then for every $b \in B$ there exists $b' \in B'$ such that $(B - \{b\}) \cup \{b'\}$ is a basis.
 - (1) *Proof*. This, too, uses the exchange lemma.
 - viii. *Corollary*. All bases are equinumerous.

- (1) The proof in my notes uses the axiom of choice to well-order the bases, then uses recursion on the order relations, with the exchange theorem, to construct a bijection from one base to the other.
 - (2) I suspect that this can be shortcut to use Zorn's lemma instead of well-ordering and recursion. That would be a good topic for the writer to investigate.
 - (3) This would not eliminate dependence on the axiom of choice, because we've already used Zorn's lemma, which is equivalent to that axiom.
- d. *Dimension*
- i. *Definition.* The dimension of a vector space is the common cardinality of its bases.
 - ii. *Corollary.* $\dim(X \cap Y) + \dim(X \vee Y) \leq \dim X + \dim Y$ for any subspaces X, Y .
 - (1) *Proof.* This comes from repeated use of the theorem about extending independent subsets to bases, then counting.
 - (2) My notes are set in a context more general than vector spaces. For vector spaces, equality may hold here. I'm not sure.
 - (3) My notes give some conditions under which equality does hold. I think they are met in the finite-dimensional case. Investigating the infinite dimensional case would be a good project for the writer.
 - iii. *Rank & nullity theorem*
 - (1) My notes provide a general proof that when X is a subspace of a space Y , then $\dim X + \dim(Y/X) = \dim Y$, where Y/X is, at least, closely related to the usual quotient space.
 - (2) I think that leads in the finite dimensional case to the theorem, if f is a linear map from one space V to another, then $\dim V = \dim \text{Range } f + \dim \text{Kernel of } f$.
 - (3) I don't know whether precisely that theorem holds in infinite dimensional linear algebra. Investigating that would be a good project for the writer.
- e. *Examples.* Editing shifted this item down probably far later than it should be.
- i. Finite dimensional
 - ii. Infinite dimensional
- f. ??What features of the definition of "vector space" are really required for this theory?
- i. We can accommodate fields, perhaps noncommutative, of any characteristic. We don't want to do modules over rings.
 - ii. The train of theorems presented above really only depends on the most basic notions of linear algebra, plus the exchange lemma. That context can be formalized in several ways. If that is done, then the results will be applicable in a much broader context. We may want to include a paragraph about that.

g. *References*

Baer, Reinhold. 1952. *Linear Algebra and Projective Geometry*. Pure and applied mathematics, 2. New York: Academic Press.

Van der Waerden, B. L. [1937] 1953. *Modern Algebra*, in part a development from lectures by E. Artin and E. Noether, volume 1. Translated from the second German edition by Fred Blum, with revisions and additions by the author. Revised English edition. New York: Frederick Ungar Publishing Co. Originally published as *Moderne Algebra* in Berlin, 1931, by Julius Springer.