

1. *Assignment*
 - a. Continue formulating questions about the paper we're outlining in class.
 - b. Continue formulating questions about the social organization of mathematics.
 - c. I've collected a few examples to illustrate some points I want to make. The first one, hypertext, starts out in a didactic fashion, about openings. But for me, it's really about finding a story line. It's a wonderful example of using that to tie together a paper, with several surprises along the way. I've listed it in the course bibliography as Greenblatt 2007. It's available online:
<http://harvardmagazine.com/2007/09/p4-writing-as-performance.html>
2. *In-class outline*
 - a. Isn't there a theorem about expanding an independent set, or contracting a spanning set, to form a basis? Shouldn't it go early in the topics?
 - b. Consider more major topics, including generalization beyond linear algebra. I did that, following a paper of this sort that I wrote while a graduate student.
3. Here is the current state of the outline.
 - a. *Introduction*
 - i. ??Should we explain the basics?
 - (1) vector space
 - (2) independent subset
 - (3) spanning subset
 - ii. The idea of the following arrangement is to base all the initial results on the ideas in the introduction, and on the following
 - b. *Exchange Lemma*. If a vector p is not in the span of a set X of vectors, but is in the span of $X \cup \{q\}$ for some vector q , then q is in the span of $X \cup p$.
 - i. *Proof*. p would be the sum of a linear combination of vectors in X and uq for some scalar u . But u must be nonzero, so has an inverse, and q would thus be a linear combination of vectors in $X \cup \{p\}$.
 - c. *Theorem*. The following properties of a set B of vectors are equivalent:
 - i. B is independent and span the whole vector space V .
 - ii. B is a maximal independent subset of V .
 - iii. B is a minimal spanning subset of V .
 - d. *Definition*. A subset with these properties is called a basis.
 - e. *Theorem*. Every independent set X of vectors is contained in a basis.
 - i. *Proof*. Show that the family $\mathcal{F}$ of all independent sets containing X satisfies the hypotheses of Zorn's lemma. That takes only a line or two. Zorn's lemma then implies that $\mathcal{F}$ has a maximal member.
 - f. *Corollary*. Every vector space has a basis.
 - i. *Proof*. The empty family is independent.
 - g. *Theorem*. Every set X of vectors contains a basis of the span of X .
 - i. *Proof*. This is a little more complicated, and uses the exchange lemma.

- h. *Exchange Theorem.* If B, B' are bases then for every $b \in B$ there exists $b' \in B'$ such that $(B - \{b\}) \cup \{b'\}$ is a basis.
 - i. *Proof.* This, too, uses the exchange lemma.
- i. *Corollary.* All bases are equinumerous.
 - i. The proof in my notes uses the axiom of choice to well-order the bases, then uses recursion on the order relations, with the exchange theorem, to construct a bijection from one base to the other.
 - ii. I suspect that this can be shortcut to use Zorn's lemma instead of well-ordering and recursion. That would be a good topic for the writer to investigate.
 - iii. This would not eliminate dependence on the axiom of choice, because we already used Zorn's lemma, which is equivalent to that axiom.
- j. *Definition.* The dimension of a vector space is the common cardinality of its bases.
- k. *Corollary.* $\dim(X \cap Y) + \dim(X \vee Y) \leq \dim X + \dim Y$ for any subspaces X, Y .
 - i. *Proof.* This comes from repeated use of the theorem about extending independent subsets to bases, then counting.
 - ii. My notes are set in a context more general than vector spaces. For vector spaces, equality may hold here. I'm not sure.
 - iii. My notes give some conditions under which equality does hold. I think they are met in the finite-dimensional case. Investigating the infinite dimensional case would be a good project for the writer.
- l. *Rank & nullity theorem*
 - i. My notes provide a general proof that when X is a subspace of a space Y , then $\dim X + \dim(Y/X) = \dim Y$, where Y/X is, at least, closely related to the usual quotient space.
 - ii. I think that leads in the finite dimensional case to the theorem, if f is a linear map from one space V to another, then $\dim V = \dim \text{Range } f + \dim \text{Kernel of } f$.
 - iii. I don't know whether precisely that theorem holds in infinite dimensional linear algebra. Investigating that would be a good project for the writer.
- m. *Examples.* Editing shifted this item down probably far later than it should be.
 - i. Finite dimensional
 - ii. Infinite dimensional
- n. ??What features of the definition of "vector space" are really required for this theory?
 - i. We can accommodate fields, perhaps noncommutative, of any characteristic. We don't want to do modules over rings.
 - ii. The train of theorems presented above really only depends on the most basic notions of linear algebra, plus the exchange lemma. That context can be formalized in several ways. If that is done, then the results will

be applicable in a much broader context. We may want to include a paragraph about that.

o. *References*

Baer, Reinhold. 1952. *Linear Algebra and Projective Geometry*. Pure and applied mathematics, 2. New York: Academic Press. [Cited in ??.]