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1. *Assignment*
 - a. Continue formulating questions about the social organization of mathematics.
 - b. Continue formulating questions about the paper we're outlining in class.
 2. *Topics for in-class outline of an expository paper.* We considered
 - a. dependence and independence questions in linear algebra
 - b. the fundamental theorem of finitely generated Abelian groups
 - c. the fundamental theorem of arithmetic
 - d. Cauchy's theorems in complex analysis.We chose the first.
 3. *Outline of expository paper.* I'll keep a *WordPerfect* document posted with the current state of the outline, and edit it live, according to class discussions. After each discussion I'll include a "snapshot" of its current state in the corresponding class outline. Here's the first snapshot.
 - a. Introduction
 - b. ??Should we explain the basics?
 - c. Rank & nullity theorem
 - d. Extension of the theory to infinite-dimensional cases
 - e. ??What features of the definition of "vector space" are really required for this theory?
 - f. References
 4. The introduction and references came first. (It really helps to establish some required parts first, to avoid the "blank page" hurdle.) Then the rank & nullity item, to establish at least one major topic.
 5. I'll use ?? to signal a question about something that might go at the place designated. The question about basics came up last year and caused much discussion. Let's count on doing that, but not until we get more major items down, so we have an idea of what the basics might be.
 6. The second major item appeared next, probably because it lies near one student's own chosen topic. This led me to explain a little about the finite- and infinite-dimensional cases.
 - a. You're used to seeing an element of a vector space V called *dependent* on a subset $S \subseteq V$ if it can be written as a finite sum of scalar multiples of elements of S . That notion is not sensitive to that of the dimension of V .
 - b. You're used to classes that require V to be *finite-dimensional*: it should contain a finite subset S on which every vector in V is dependent. From that assumption, you derive several rather major theorems about bases and dimension.

- c. Can those theorems and the related definitions be formulated and proved without that hypothesis? Yes, for the most part. But you must allow the dimension of V to be an *infinite* cardinal.
 - d. Is that a useful extension of the theory? To some extent. It is useful in studying dimensional aspects of the foundations of geometry, where you want to isolate the places where dimension occurs in the basic notions. What better than to simply banish it from the assumptions? The infinite-dimensional theory is also useful in analyzing some basic notions in higher algebra. But at least until quite recently, this notion has not seen much use in analysis. I've seen a few papers and one monograph in that area.
 - e. In analysis there's a competing notion of dependence that works for some vector spaces—for example, Hilbert space—that support a notion of convergent series. In such a space V , a vector might be called *dependent* on a subset $S \subseteq V$ if it can be written as the sum of a convergent infinite series of scalar multiples of elements of S . Much of the theory of bases and dimension can be modified to fit this situation, and the result is very useful in many application areas.
 - f. But the results for the two sets of definitions may be different. For example, the dimension of a space is defined to be the cardinal of a minimal set S on which all vectors in V depend. For the finite-sum definition of dependence, the dimension of Hilbert space is 2^ω , the cardinal of the set of all real numbers. But for the convergent-infinite-series definition, the dimension is just ω , the cardinal of the set of all natural numbers.
7. The second question arose from an ill-formed inquiry phrased using the word *independence*. I interpreted it to refer to the question whether portions of the theory in question are really dependent on all the other requirements for vector spaces.
- a. You're used to seeing vector-space theory developed with real scalars only, then extended by noting that nothing depended on that requirement, and a comparable theory exists for complex scalars. How about relaxing that requirement further? Many texts present the theory with an arbitrary field of scalars. But you learn in algebra classes about structures that are almost fields, but for which the commutative law fails—*noncommutative fields*, for example the quaternions. (*Fields*, commutative or not, are also called *division rings* or *skew fields*.) Can vector-space theory be developed for scalars that form even a noncommutative field?
 - b. Yes, indeed. Reinhold Baer's *Linear Algebra and Projective Geometry* develops the entire theory for arbitrary, finite or infinite dimension, over arbitrary, commutative or skew fields. I'll bring the book next time so that we can construct a reference to it.
 - c. This tale of extensions to more and more general scalars can stretch yet further. Do you know where it goes?