

1. *This class begins at 0800!*
2. *Due today*
 - a. Term papers!
 - b. All remaining homework.
 - i. That includes all attempts that have been returned but not finished. Turn in all work, including my earlier comments. That is the way I determine partial credit.
 - c. Snail-mail address so that I can return stuff to you.
 - i. I will return homeworks and my comments about your term papers.
 - ii. I won't return the term paper itself, unless you included something irreplaceable therein, in which case I'll attempt to photocopy that. So you should keep copies of your papers, in order to make sense of my comments.
 - d. Email address if you're graduating, and thus losing an `sfsu.edu` address that I've been using to reach you.
3. I expect to remain accessible by email, except for some periods when I'm traveling. You have my email address. Except for a couple of brief visits, I don't expect to be in San Francisco much until November.
4. *Student reports.* The comments after each of the following capsules are mine, uttered following each report. I'm not attempting to summarize the reports.
5. *Ms. Morgan* presented a report on her project about measurement theory.
 - a. This theory focuses on requirements on a system of judgments about objects, such as whether one is larger than, or at most twice as long as, another, to insure the existence of a mapping from the objects to real numbers that preserves the outcomes of these judgments—in these cases, compatible assignments of sizes or lengths to the objects.
 - b. While this may seem rather simple, consider the problems that arise in devising voting schemes that judge the merit of propositions or candidates. Theory shows that almost any such scheme other than direct polling of all voters can produce unintended results. Indirect schemes are necessary both in politics and in science, hence the great interest in measurement theory.
 - c. The earliest examples of the kind of mathematics used in measurement theory are the nineteenth-century studies of continuity that relate geometric objects to real numbers.
6. *Mr. Krogh-Freeman* presented a report on his project about Turing-computable and general recursive functions.
 - a. The two types of recursive definitions allowed for general recursive functions correspond to two types of recursive proof used in this course: recursion on the natural numbers and on well-ordered sets. But in this context, they're only applied to natural numbers: one emphasizes the successor function $n \rightarrow$

- $n + 1$; the other, the order relation. It is known that some general recursive functions cannot be defined using only the first type of recursion.
- b. Turing-computable functions were introduced during the 1930s. By then, IBM and Remington-Rand punched-card data-processing equipment had been in use for several decades, but the stored-program computers organized along the lines of Turing's definition wouldn't appear for another five to ten years.
7. *Ms. Crump* presented a report on her project about intuitionistic and constructive logic.
 - a. L. E. J. Brouwer (equally famous for a fixed-point theorem in analysis, and for work in combinatorial topology) proposed a drastic restriction in the rules of logic as a way to avoid contradictions in mathematics. In particular, he regarded negation and disjunction very differently from classical logicians. His work was cast informally, but he agreed that a formal presentation by his student Arend Heyting captured its essence.
 - b. Using Heyting's formalization, Gödel showed that for every proposition p provable in classical logic, the proposition $\neg\neg p$ was intuitionistically provable. Thus intuitionism is not as drastic a departure from classical logic as had originally been supposed.
 - c. Constructivism, a more recent development, is a much stronger restriction.
 8. *Ms. Thompson* presented a report on her project about the Quine–Jensen system NFU of set theory.
 - a. Quine's NF permits use of a universal set and of complements of sets, whereas in ZF set theory no set is universal and only relative complements are allowed. To compensate for this additional freedom and avoid Russell's antinomy, Quine required that the formulas mentioned in the separation axiom be *stratified*. Since the formula $x \in x$ is not stratified, the separation axiom does not lead to the antinomy.
 - b. But Specker showed that the axiom of choice is inconsistent with NF. Then Jensen proposed a modified theory NFU in which there are objects—*Ur-elemente*—that have no elements but are different from the empty set. This theory is in fact consistent with the axiom of choice.
 9. *Mr. Rizzolo* presented a report on his project about forcing, the method Cohen introduced to prove the independence of the continuum hypothesis.
 - a. He emphasized the level of difficulty. The only simplifications I knew about earlier involved use of measure theory beyond that covered at SFSU. (That is, all the difficulty is concentrated in that one area; the logical part of the argument is simple.) However, Mr. Rizzolo has found a recent presentation that is closer to Cohen's original argument and seems simpler to me. I'll have to study it!
 10. *Ms. Evans* presented a report on her project about different means of introducing the real number system, given the rational number system.
 - a. Cantor's method applies standard arguments to Cauchy sequences of rational numbers, so that real numbers can be regarded as equivalence classes of Cauchy sequences of rational numbers.

- b. Dedekind's method applies standard least-upper- and greatest-lower-bound arguments to partitions of the rational numbers into two sets, the members of one of which are all smaller than all the members of the other. In this approach, the real numbers can be regarded as these partitions.
 - c. The main difficulty in Cantor's approach is handling the equivalence of Cauchy sequences: equivalent sequences must behave as though they have the same limit, but must be considered in detail before the limit itself can be mentioned.
 - d. The main difficulty in Dedekind's approach is developing the properties of multiplication of real numbers.
 - e. Cantor's approach generalizes to provide a means of completing metric spaces. Dedekind's generalizes to provide a means of completing lattices.
 - f. There are at least two methods rather different from these, which, however, do not generalize, and thus are rarely mentioned in mathematics courses.
11. *Ms. Sanford* presented a report on her project about the work of Cantor leading to the diagonal argument.
- a. She emphasized the hurdles he had to overcome to get his theories accepted.
 - b. This is still a research area: now and again correspondence comes to light that provides new information about how leading mathematicians and philosophers felt about these matters.