

1. Let Y be a set of variables of an elementary language. Define the phrase, “formula q differs from formula p at most by alphabetic change of bound variables to variables in Y ” recursively as follows:
 - a. If no quantifier occurs in p , then q is the same as p .
 - b. If p is the formula $\forall x p'$ for some variable x and some formula p' , then q is p or $\forall y p'_y$ for some variable $y \in Y$ that does not occur in p' .
2. *Lemma.* Let p be a formula, t be a term, x be a variable, and Y be an infinite set of variables that do not occur in p . Then there is a formula q that differs from p at most by alphabetic change of bound variables to variables in Y , such that $p \in c(\{q\})$, $q \in c(\{p\})$, and t is substitutable for x in q . *Proof,* by recursion on the length of p . (This is another place where elementary logic requires an infinite set of variables.)
 - a. If no quantifier occurs in p let q be the same as p .
 - b. Let p be n symbols long, and suppose the result is true for all formulas p of shorter length.
 - c. If p is $\neg p'$ for some formula p' shorter than p , then there is a formula q' that differs from p' at most by alphabetic change of bound variables to variables in Y , such that $p' \in c(\{q'\})$, $q' \in c(\{p'\})$, and t is substitutable for x in q' . Let q be $\neg q'$. Then q differs from p at most by alphabetic change of bound variables to variables in Y , and t is substitutable for x in q . Moreover, $p \in c(\{q\})$ and $q \in c(\{p\})$ because the following formulas are tautologous:

$$(p \Rightarrow q) \Rightarrow (\neg q \Rightarrow \neg p)$$

$$(\neg q \Rightarrow \neg p) \Rightarrow (p \Rightarrow q).$$
 - d. If p is $(p' \Rightarrow p'')$ for some formulas p', p'' shorter than p , then there is a formula q' that differs from p' at most by alphabetic change of bound variables to variables in Y , such that $p' \in c(\{q'\})$, $q' \in c(\{p'\})$, and t is substitutable for x in q' . Let Y' be the set of all variables in Y that do not occur in q' . Since Y' is infinite, there is a formula q'' that differs from p'' at most by alphabetic change of bound variables to variables in Y' , such that $p'' \in c(\{q''\})$, $q'' \in c(\{p''\})$, and t is substitutable for x in q'' . Let q be $(q' \Rightarrow q'')$. Then q differs from p at most by alphabetic change of bound variables to variables in Y and t is substitutable for x in q . Moreover, $p \in c(\{q\})$ and $q \in c(\{p\})$ because the following formulas are tautologous:

$$(q' \Rightarrow p') \Rightarrow ((p'' \Rightarrow q'') \Rightarrow ((p' \Rightarrow p'') \Rightarrow (q' \Rightarrow q'')))$$

$$(p' \Rightarrow q') \Rightarrow ((q'' \Rightarrow p'') \Rightarrow ((q' \Rightarrow q'') \Rightarrow (p' \Rightarrow p''))).$$
 - e. If p is the formula $\forall y p'$ for some variable y and some formula p' shorter than p , then there is a formula q' that differs from p' at most by alphabetic change of bound variables to variables in Y , such that $p' \in c(\{q'\})$, $q' \in c(\{p'\})$, and t is substitutable for x in q' . Because Y is infinite, there

is a variable $z \in Y$ that differs from x and does not occur in p' nor t . Let q be $\forall z p'_z{}^y$. Then q differs from p at most by alphabetic change of bound variables to variables in Y and t is substitutable for x in q .

- i. Since z doesn't occur in p' , it's substitutable for y in p' , $\forall y p' \Rightarrow p'_z{}^y$ is an axiom, and thus $p'_z{}^y \in c(\{\forall y p'\})$. The generalization principle yields $\forall z p'_z{}^y \in c(\{\forall y p'\})$, since z doesn't occur in $\forall y p'$. That is, $q \in c(\{p\})$.
 - ii. Since z doesn't occur in p' and differs from y , $\forall z$ is not a substring of $p'_z{}^y$, and thus y is substitutable for z in $p'_z{}^y$; moreover, $p'_{z,y}{}^z = p'$. Thus $\forall z p'_z{}^y \Rightarrow p'$ is an axiom, and $p' \in c(\{\forall z p'_z{}^y\})$. The generalization principle yields $\forall y p' \in c(\{\forall z p'_z{}^y\})$, since y has no free occurrence in $\forall z p'_z{}^y$. That is, $p \in c(\{q\})$.
3. The hypothesis of the Gödel completeness theorem is that S be a set of formulas in a language L and f a formula of L such that $f \notin c(S)$. As for the Boolean theorem, this proof begins by extending the theory T to a maximal theory T^* for which $f \notin T^*$: any larger theory must contain f . But that situation is not sufficient for the Gödel theorem: it doesn't provide enough constants to construct the relational system in which f should be false. That fault is repaired as follows.
- a. By the Henkin–Hasenjäger theorem there's a conservative Henkin extension H of T in a language L^* obtained from L , such that the alphabets of L and L^* are equinumerous.
 - b. $f \notin H$ because $f \in H$ would imply $f \in T$ since f is a formula of L and H is a conservative extension of T .
 - c. Use Lindenbaum's theorem to extend H to a maximal theory T^* in L^* for which $f \notin T^*$: any larger theory in L^* must contain f .
 - d. *Lemma.* T^* is a Henkin theory. *Proof.* Suppose p is a formula of L^* , x is a variable, and $(\exists x p) \in T^*$. Since H is a Henkin theory in that language, there is a constant c of L^* such that $(\exists x p \Rightarrow p_c^x) \in H \subseteq T^*$. Thus $p_c^x \in T^*$.
 - e. You can imitate the last step in the proof of the Boolean completeness theorem to show that $p \in T^*$ if and only if $\neg p \notin T^*$, for every formula p of L .
4. The conclusion of the Gödel completeness theorem is that there be a relational system compatible with L and a valuation φ of L in it such that $\varphi(s)$ is true for each $s \in S$ but $\varphi(f)$ is false. That system is constructed in two stages. First, let M be the set of all terms of L^* . Make M into a relational structure compatible with L by defining associated constants, operators, and relations as follows:
- a. Each constant of L should be an associated constant of M .
 - b. For each n -ary operator o of L associate with M the n -ary operator that maps terms $t_1, \dots, t_n$ to the term consisting of o concatenated with $t_1, \dots, t_n$.
 - c. For each n -ary predicate R of L except equality, associate with M the n -ary relation that holds of terms $t_1, \dots, t_n$ if and only if the formula consisting of R concatenated with $t_1, \dots, t_n$ is in T^* .
5. Define a binary relation $\equiv$ on M by setting $t \equiv u$ if and only if $(t = u) \in T^*$. The equality axioms were chosen to insure that $\equiv$ is a congruence for all operators and relations associated with M . That is, $\equiv$ is an equivalence on M and

- a. $t_1 \equiv u_1 \ \& \ \cdots \ \& \ t_n \equiv u_n \Rightarrow o t_1 \cdots t_n \equiv o u_1 \cdots u_n$
- b. $t_1 \equiv u_1 \ \& \ \cdots \ \& \ t_n \equiv u_n \Rightarrow (R t_1 \cdots t_n \Leftrightarrow R u_1 \cdots u_n)$
for any n -ary operator o and any n -ary relation R associated with M and any $t_1, u_1, \dots, t_n, u_n \in M$.
6. The second construction stage is to define the quotient relational system $M/\equiv$ as usual in algebra:
 - a. The elements of M^* are the equivalence classes $c/\equiv$ for all $c \in M$.
 - b. Each n -ary operator o associated with M corresponds to an n -ary operator $o/\equiv$ associated with $M/\equiv$ such that for all $t_1, \dots, t_n \in M$, $o/\equiv$ maps $\langle t_1/\equiv, \dots, t_n/\equiv \rangle$ to $o(t_1, \dots, t_n)/\equiv$.
 - c. Each n -ary relation R associated with M corresponds to an n -ary relation $R/\equiv$ associated with $M/\equiv$ such that for all $t_1, \dots, t_n \in M$, $R/\equiv$ holds for $t_1/\equiv, \dots, t_n/\equiv$ just when R holds for $t_1, \dots, t_n$.
 This quotient system $M/\equiv$ is compatible with L .
7. The desired valuation φ is easy to define: $\varphi(v) = v/\equiv$ for each variable v . Extending the domain of φ to the set of all terms yields $\varphi(t) = t/\equiv$ for each term t .
8. Here is the main argument of the Gödel completeness theorem: *for any formula p of L , $\varphi(p)$ is true if and only if $p \in T^*$. Proof* by recursion on the length of p .
 - a. Suppose p is $(t = u)$ for some terms t, u , so that $\varphi(t) = t/\equiv$ and $\varphi(u) = u/\equiv$. Then $\varphi(p)$ is true if and only if $\varphi(t) = \varphi(u)$, and thus if and only if $t/\equiv = u/\equiv$. That condition is equivalent to $t \equiv u$, which by definition of $\equiv$ holds if and only if $(t = u) \in T^*$: that is, if and only if $p \in T^*$.
 - b. Suppose p is $R t_1 \cdots t_n$ for some n -ary predicate R of L and some terms $t_1, \dots, t_n$. Then $\varphi(p)$ is true if and only if the relation corresponding to R associated with M holds for $\varphi(t_1), \dots, \varphi(t_n)$: that is, if and only if $R/\equiv$ holds for $t_1/\equiv, \dots, t_n/\equiv$. That condition is equivalent to $R t_1 \cdots t_n \in T^*$, which is the same as saying $p \in T^*$.
 - c. Suppose p is $\neg q$ for some formula q . Because that's shorter than p , $\varphi(q)$ is true if and only if $q \in T^*$. Thus, $\varphi(p)$ is true if and only if $\varphi(q)$ is false, hence if and only if $q \notin T^*$, hence if and only if $\neg q \in T^*$, which is to say $p \in T^*$.
 - d. Suppose p is $(q \Rightarrow r)$ for some formulas q, r . Because those are shorter than p , $\varphi(q)$ is true if and only if $q \in T^*$ and $\varphi(r)$ is true if and only if $r \in T^*$. Thus, $\varphi(p)$ is false if and only if $\varphi(q)$ is true and $\varphi(r)$ false, hence if and only if $q \in T^*$ and $r \notin T^*$, hence if and only if $q \in T^*$ and $\neg r \in T^*$. This condition is equivalent to $\neg(q \Rightarrow r) \in T^*$ because formulas $\neg(q \Rightarrow r) \Rightarrow \neg q$, $\neg(q \Rightarrow r) \Rightarrow r$, and $\neg q \Rightarrow (r \Rightarrow \neg(q \Rightarrow r))$ are tautologous. In summary, $\varphi(p)$ is false if and only if $\neg(q \Rightarrow r) \in T^*$, hence if and only if $(q \Rightarrow r) \notin T^*$, which is to say $p \notin T^*$.
 - e. Suppose p is $\forall x q$ for some variable x and some formula q .
 - i. Suppose $\varphi(p)$ is true. Since T^* is a Henkin theory, there's a constant c in its language L^* such c is substitutable for x in q and the formula $(\exists x \neg q \Rightarrow \neg q_c^x) \in T^*$. Let χ be the valuation such that $\chi(v) = \varphi(v)$ for all variables v except that $\chi(x) = c$. Then $\varphi(q_c^x)$ —that is, $\chi(q)$ —is true, so $q_c^x \in T^*$ since it is shorter than p . This implies $\neg q_c^x \notin T^*$, and

- therefore $(\exists x \neg q) \notin T^*$ because $(\exists x \neg q \Rightarrow \neg q_c^x) \in T^*$. In turn, that implies $(\neg \exists x \neg q) \in T^*$, which is to say $p \in T^*$.
- ii. Conversely, suppose $\varphi(p)$ is false. Then $\chi(q)$ is false for some valuation χ such that $\chi(v) = \varphi(v)$ for all variables v except possibly x . Let $\chi(x) = c$.
- (1) If c is substitutable for x in q , then $\chi(q) = \varphi(q_c^x)$, and therefore $q_c^x \notin T^*$ because it is shorter than p . Since $p \Rightarrow q_c^x$ is an axiom, $p \notin T^*$.
 - (2) If c is not substitutable for x in q , it must be a variable that occurs in the scope of a quantifier $\forall x$ in q . The lemma in the second item of the current outline shows how to find a formula q' that differs from q only in the choice of bound variables, such that c is substitutable for x in q' and $\forall x q' \in c(\{\forall x q\})$. It is easy to see that $\chi(q) = \chi(q') = \varphi(q_c^x)$. Then $q_c^x \notin T^*$ because it is shorter than p , and $(\forall x q') \notin T^*$ because $\forall x q' \Rightarrow q_c^x$ is an axiom, hence finally $(\forall x q) \notin T^*$, which is as much as to say $p \notin T^*$.
9. That concludes the proof of the Gödel completeness theorem: if S is a set of formulas of an elementary language L and formula $f \notin c(S)$, then with the valuation φ in the relational system $M/\equiv$, $\varphi(s)$ is true for every $s \in S$ but $\varphi(f)$ is false, because $S \subseteq T^*$ and $f \notin T^*$.
- a. Moreover, the cardinality of $M/\equiv$ does not exceed the sum of the cardinals of the sets of constants, variables, and operators of L .
 - b. Although the axiom of choice was used in this proof, it is not required if that sum is ω .
 - c. The theorem is often phrased slightly differently: *if S is a set of sentences of L and formula $f \notin c(S)$ then S has a model in which f is not true.* Proof. in a given relational system a sentence is either true for all valuations or false for all valuations.
10. A theory T in a language L is called *consistent* if it is different from the set of all sentences. It is easy to see that the following conditions are equivalent:
- a. T is consistent,
 - b. T does not contain both f and $\neg f$ for any formula f of L ,
 - c. T has a model.
11. *Compactness theorem.* Because the syntactic and semantic closure operators on the set of all formulas agree, and the syntactic operator is finitary, so is the semantic operator: if S is a set of formulas then any formula that is true in every model of S is true in every model of some finite subset of S .
- a. As a familiar application of this theorem, consider the elementary language L and axiom system of the theory T of fields of characteristic zero. It contains axioms $1+1 \neq 0, 1+1+1 \neq 0, 1+1+1+1 \neq 0, \dots$. If a sentence s in this language is true in every field of characteristic zero, then there must be a positive integer p such that s is true in every field of finite characteristic greater than p .
12. With only one inference rule, this presentation of elementary logic avoids both the generalization and *existential instantiation* rules.

- a. The former was discussed earlier.
- b. The latter consists in assuming or deriving an existential statement $\exists x p$, reserving the variable x to stand for something such that p , then reasoning about x .
- c. Example: derive $\exists x(x \cdot x = 1+1+1)$ then reserve x to designate such a number.
- d. Formalizing this rule is difficult because use of the instantiation variable is thereafter restricted. In this example, you can prove $x > 1$ but cannot then generalize to $\forall x(x > 1)$.
- e. Nevertheless, the technique is used everywhere, all the time.
- f. In the proof of Gödel's theorem, the main difficulty is to implement a version of instantiation!
- g. In a Berkeley lecture not long before his recent death Paul Cohen remarked that logicians don't in general understand instantiation very well, and some progress in our subject awaits a really deep study of it.