

1. The next two results compensate for the omission of the rule of generalization from the inference rules.
 - a. *Lemma.* If the variable x does not occur free in any member of a set S of formulas, then $c(S) \subseteq \{p : (\forall x p) \in c(S)\}$. *Proof.*
 - i. All elementary axioms belong to the set R on the right.
 - ii. If $p \in S$ then $p \in c(S)$, and since $p \Rightarrow \forall x p$ is an axiom, it follows that $(\forall x p) \in c(S)$, hence $p \in R$.
 - iii. If p, q are elementary formulas and $(p \Rightarrow q) \in R$, then $\forall x(p \Rightarrow q) \in c(S)$, hence $(\forall x p \Rightarrow \forall x q) \in c(S)$ because $\forall x(p \Rightarrow q) \Rightarrow (\forall x p \Rightarrow \forall x q)$ is an axiom. Thus if also $p \in R$, then $(\forall x p) \in c(S)$, hence $(\forall x q) \in c(S)$, and thus finally $q \in R$.
 - iv. The desired result follows because $c(S)$ is the smallest set with properties (i–iii).
 - b. *Theorem (generalization rule).* If the variable x does not occur free in any member of a set S of formulas and $p \in c(S)$, then $(\forall x p) \in c(S)$.
2. The next two results explore the degree to which constants and variables play similar roles. Essentially, they say that if you deduce a property of a constant c without relying on any property of c stated in your premisses, then your proof can be restated as a general one, with any object in the underlying set in place of the constant.
 - a. *Lemma.* Let S be a set of formulas, $p \in c(S)$, and c be a constant with no occurrence in any member of S . Then there is a variable v , with no occurrence in p , for which $(\forall v p_v^c) \in c(S)$ —here, p_v^c is the result of substituting an occurrence of v for each occurrence of c in p . *Proof.*
 - i. There is a deduction of p from some finite subset $E \subseteq S$; select any variable v with no occurrence in any formula in E or in that deduction. (*This is why elementary languages must have infinitely many variables!*)
 - ii. Substitute v for c everywhere in that deduction. Check that
 - (1) this changes every axiom into another axiom;
 - (2) it doesn't change S ;
 - (3) if it changes formulas q, r to q', r' , then it changes $q \Rightarrow r$ to $q' \Rightarrow r'$.
 - iii. Thus the result is a deduction of p_v^c from $E \subseteq S$.
 - iv. Since v doesn't occur in any member of E , the generalization rule yields $(\forall v p_v^c) \in c(E) \subseteq c(S)$.
 - b. *Lemma.* Let S be a set of formulas, p a formula, x a variable, and c be a constant with no occurrence in p or any member of S . If $p_c^x \in c(S)$, then $(\forall x p) \in c(S)$. *Proof.*
 - i. By the previous lemma there exists a variable v with no occurrence in p_c^x such that $\forall v (p_c^x)_v^c \in c(S)$.
 - ii. That is, $(\forall v p_v^x) \in c(S)$, because c does not occur in p .

- iii. $\forall v p_v^x \Rightarrow \forall x \forall v p_v^x$ is an axiom because x has no free occurrence in $\forall v p_v^x$.
 - iv. By (ii, iii) and modus ponens, $(\forall x \forall v p_v^x) \in c(S)$.
 - v. Because v does not occur in p , x is substitutable for v in p_v^x and $\forall v p_v^x \Rightarrow (p_v^x)_x^v$ is an axiom.
 - vi. But $(p_v^x)_x^v = p$, hence $\forall v p_v^x \Rightarrow p$ is an axiom.
 - vii. Its generalization $\forall x (\forall v p_v^x \Rightarrow p)$ is, too.
 - viii. And so is $\forall x (\forall v p_v^x \Rightarrow p) \Rightarrow (\forall x \forall v p_v^x \Rightarrow \forall x p)$.
 - ix. By (vii, viii) and modus ponens, $(\forall x \forall v p_v^x \Rightarrow \forall x p) \in c(S)$.
 - x. By (iv, ix) and modus ponens, $(\forall x p) \in c(S)$.
3. A *conservative* extension of a theory T in an elementary language L is a theory $T' \supseteq T$ in a language $L' \supseteq L$ such that the only theorems of T' in which only symbols of L occur are the theorems already in T . When you extend a theory conservatively by adding to its language and consequently to its theorems, you gain no new theorems in the original language. All new theorems involve new language features.
4. Consider this process for extending a theory T in an elementary language L .
- a. Corresponding to the sentences of L of the form $\exists x p$ construct distinct constants $c_{\exists x p}$, different from all the symbols used by L . Add these to L to get a new language $L^\exists$.
 - b. Let C be the set of all the corresponding formulas $\exists x p \Rightarrow p_{c_{\exists x p}}^x$ and $T^\exists = c(T \cup C)$.
 - c. If the alphabet of L has cardinal λ , then the set of its strings also has cardinal λ , so the cardinal of the alphabet of $L^\exists$ is $\lambda + \lambda = \lambda$. (This cardinal arithmetic computation does not depend on the axiom of choice if $\lambda = \omega$.)
5. *Theorem.* $T^\exists$ is a conservative extension of T in a language whose alphabet is equinumerous with that of L . *Proof.*
- a. Suppose s is a sentence in $T^\exists$ in which occur only symbols of L .
 - b. To show: $s \in T$.
 - c. By (a) there's a deduction of s from $T \cup \{h_1, \dots, h_n\}$, where each h_k is a sentence of $L^\exists$ of the form $\exists x p \Rightarrow p_{c_{\exists x p}}^x$ for some variable x , some formula p of L , and the constant $c_{\exists x p}$ does not occur in any formula of L .
 - d. By the deduction theorem, there is a deduction from T of the sentence $g = (h_1 \Rightarrow (h_2 \Rightarrow \dots (h_n \Rightarrow s) \dots))$. I shall show that if a sentence $(h_k \Rightarrow e) \in T$, then $e \in T$. Applying that result n times will yield $s \in T$, as desired.
 - i. First, $\neg h_k \in c(T \cup \{\neg e\})$, because the formula $(h_k \Rightarrow e) \Rightarrow (\neg e \Rightarrow \neg h_k)$ corresponds to a Boolean tautology. Recall that $\neg h_k = \neg(\exists x p \Rightarrow p_{c_{\exists x p}}^x)$.
 - (1) It follows that $(\neg p_{c_{\exists x p}}^x) \in c(T \cup \{\neg e\})$ because the formula $\neg(\exists x p \Rightarrow p_{c_{\exists x p}}^x) \Rightarrow \neg p_{c_{\exists x p}}^x$ corresponds to a Boolean tautology. By the previous lemma, $(\forall x \neg p) \in c(T \cup \{\neg e\})$.
 - (2) It also follows that $(\exists x p) \in c(T \cup \{\neg e\})$ because the formula $\neg(\exists x p \Rightarrow p_{c_{\exists x p}}^x) \Rightarrow \exists x p$ corresponds to a Boolean tautology. But $(\exists x p) = (\neg \forall x \neg p)$.
 - ii. By the deduction theorem, $(\neg e \Rightarrow \forall x \neg p), (\neg e \Rightarrow \neg \forall x \neg p) \in c(T) = T$.

- iii. Finally, $e \in T$ because the formula $(\neg e \Rightarrow \forall x \neg p) \Rightarrow ((\neg e \Rightarrow \neg \forall x \neg p) \Rightarrow e)$ corresponds to a Boolean tautology.
- 6. A theory T in an elementary language L is called a *Henkin* theory if for any sentence in L of the form $\exists x p$ there is a constant c of L such that c is substitutable for x in p and T contains $\exists x p \Rightarrow p_c^x$. You can think of a Henkin theory as one for which there are enough witnesses to substantiate every existential theorem.
- 7. *Theorem (Henkin–Hasenjäger)*. Every theory T in an elementary language L has a conservative Henkin extension in a language whose alphabet is equinumerous with that of L . *Proof*. In this proof the letters L stand for the sets of formulas of the corresponding languages.
 - a. Let $T_0 = T$, $L_0 = L$.
 - b. For each n let $T_{n+1} = T_n^\exists$, $L_{n+1} = L_n^\exists$.
 - c. Let $U = \bigcup_n T_n \supseteq T$ and M be the language you get by adding to L all the constants in all languages L_n .
 - d. Any sentence $\exists x p$ in M belongs to L_n for some n , so there's a constant c in L_n , hence in M , such that T_{n+1} , hence U , contains $\exists x p \Rightarrow p_c^x$. Thus U is a Henkin theory.
 - e. If $s \in U \cap L$, then $s \in T_n \cap L$ for some n . Choose the smallest such n . If $n > 0$, then $s \in T_{n-1}$ by the previous theorem, contradiction! Thus $n = 0$ and $s \in T_0 = T$. Therefore T is a conservative extension of T , as required.
 - f. If the set of all symbols of L has cardinal λ , then by the previous theorem the same holds for each L_k , hence also for M . This cardinal arithmetic computation does not depend on the axiom of choice if $\lambda = \omega$.
- 8. This result was reached independently in the late 1940s by Leon Henkin, in his PhD thesis written under the supervision of Alonzo Church, and by Gisbert Hasenjäger. Henkin became Tarski's closest colleague at Berkeley. He supervised our Prof. Resek's PhD work there in logic, and inspired many efforts in mathematics education and outreach. Henkin must have been fascinated by the piece of the previous argument that requires elementary languages to incorporate infinitely many variables. I took a course from him was on logics with only finitely many variables in summer 1966 at the University of Montreal. He died last year. Hasenjäger became a professor at the University of Münster in Germany. I met him there once in 1969, but know little about him. He seemed to work hard to make an impression congruent with his name!