

1. The Gödel completeness theorem—for any set S of formulas of an elementary language L , the semantic closure $c'(S)$ is included in the syntactic closure $c(S)$ —will be proved by contraposition. Given a formula f of L such that $f \notin c(S)$, it will be shown that $f \notin c'(S)$. We will construct a relational system M compatible with L and a valuation φ of L in M in which $\varphi(s)$ is true for all formulas $s \in S$ but $\varphi(f)$ is false. What should be that underlying set M ? We have few materials with which to begin the construction. In fact, we have only the language L itself. That's what we shall use, augmented by further symbols we invent when necessary. Any constants in the alphabet of L would be designed already for denoting members of M ; we'll start with them. But that's probably not enough. For example, using an elementary language for real arithmetic, with just the constants $0, 1$, we can derive $\exists x(x \cdot x = 1 + 1)$ from some conventional set S of arithmetic axioms, but there will be no constant term whose value would be the square-root of two.
2. *Axioms of elementary logic.* While I'd like to adopt logical axioms gradually as needed, that seems impractical.
 - a. First, it's useful to define recursively the notion of *generalization* of a formula:
 - i. Every formula of an elementary language is a generalization of itself.
 - ii. If x is a variable and f is a generalization of a formula g , then so is $\forall x f$.
 - b. Next, all generalizations of the Boolean axioms are elementary axioms. These are all generalizations of formulas of the form
 - i. $p \Rightarrow (q \Rightarrow p)$
 - ii. $(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r))$
 - iii. $((p \Rightarrow q) \Rightarrow p) \Rightarrow p$
 - iv. $p \Rightarrow (\neg p \Rightarrow q)$, and
 - v. $(p \Rightarrow q) \Rightarrow ((\neg p \Rightarrow q) \Rightarrow q)$,
 where p, q, r are formulas of the elementary language in use. This entails that all formulas of elementary logic that correspond to Boolean tautologies be deducible from any set of premisses.
 - c. Next, these are the elementary quantification axioms: for any formulas p, q , all generalizations of formulas
 - i. $p \Rightarrow \forall x p$, where x is a variable with no free occurrence in p ;
 - ii. $\forall x (p \Rightarrow q) \Rightarrow (\forall x p \Rightarrow \forall x q)$, for any variable x ;
 - iii. $\forall x p \Rightarrow p_y^x$, where x is a variable and y is a variable or constant substitutable for x in p .
 - d. Finally, these are the elementary axioms about equality: all generalizations of formulas
 - i. $x = x$
 - ii. $x = y \Rightarrow y = x$
 - iii. $x = y \Rightarrow (y = z \Rightarrow x = z)$

$$\text{iv. } x_1 = y_1 \ \& \ \cdots \ \& \ x_n = y_n \Rightarrow o x_1 \cdots x_n = o y_1 \cdots y_n$$

$$\text{v. } x_1 = y_1 \ \& \ \cdots \ \& \ x_n = y_n \Rightarrow (R x_1 \cdots x_n \Leftrightarrow R y_1 \cdots y_n)$$

where the x 's and y 's are variables, o is an n -ary operator of L , and R is an n -ary predicate of L .

- e. Some work minimizing this axiom set—removing axioms that can be derived from others—would be useful, but not essential.
- f. One important consequence of these axioms is the sentence $\exists x(x = x)$ —that's why the set underlying a relational structure compatible with L must be nonempty! To derive it, note that, $\forall x(x \neq x) \Rightarrow x \neq x$ is a quantification axiom, $(\forall x(x \neq x) \Rightarrow x \neq x) \Rightarrow (x = x \Rightarrow \neg \forall x(x \neq x))$ a tautology, and $x = x$ is an equational axiom.
- g. *Theorem.* All the logical axioms are true in all structures compatible with the language in use. Therefore, for any set S of formulas, the syntactic closure $c(S)$ is a subset of the semantic closure $c'(S)$.
 - i. A proof of this is a verification that the inference rule, the axioms, and the rules for extending the domain of a valuation all accurately represent common mathematical practice. You are invited to provide some of the tedious details.