

1. *Homework scores.* Let x = the number of homework points earned so far.
 - a. For two students, $0 \leq x < 12$.
 - b. For two students, $12 \leq x < 24$.
 - c. For two students, $24 \leq x < 36$.
 - d. For three students, $36 \leq x$.
2. *Endgame*
 - a. On 21 May there is 150 minutes' meeting time. There are 9 students. At 20 minutes each, you require 180 minutes to report on your term papers. One student volunteered to report on 14 May. I think we can handle the 10 minute deficit.
 - b. Term papers are due 21 May.
 - c. All homework that has not yet received full credit must be turned in on 21 May for partial credit.
 - d. From then on I will be accessible by email except for sporadic forays into the real world.
 - e. I will turn in final grades electronically before Saturday afternoon 31 May.
3. *A valuation* of an elementary language L in a compatible relational system M is a function φ from the set of variables to the set underlying the system.
 - a. The domain of φ can be extended recursively to the set of all terms, as follows:
 - i. If c is a constant of L , then $\varphi(c)$ should be the corresponding constant of M .
 - ii. If o is an n -ary operator of L and $t_1, \dots, t_n$ are terms, then $\varphi(o t_1 \dots t_n)$ should be the result of applying the corresponding operation of M to $\varphi(t_1), \dots, \varphi(t_n)$.
 - b. The domain of φ can be extended further, to the set of all formulas, by including in its range two *truth values*: *truth* and *falsehood*. (These should be any distinct things—sets, since this entire discussion is taking place within set theory—agreed on in advance.)
 - i. If t, u are terms, then $\varphi(= t u)$ should be true just when $\varphi(t) = \varphi(u)$.
 - ii. If p is an n -ary predicate of the language and $t_1, \dots, t_n$ are terms, then $\varphi(p t_1 \dots t_n)$ should be true just when the corresponding relation holds among $\varphi(t_1), \dots, \varphi(t_n)$.
 - iii. If f is a formula, then the value of the formula $\neg f$ should be true just when $\varphi(f)$ is false.
 - iv. If f, g are formulas, then $\varphi(\Rightarrow f g)$ should be true just when $\varphi(f)$ is false or $\varphi(g)$ is true.
 - v. If v is a variable and f is a formula, then the $\varphi(\forall v f)$ should be true just when $\chi(f)$ is true for every valuation such that $\psi(w) = \varphi(w)$ for all variables w different from v with free occurrences in f .

4. *Theorem.* If φ, χ are valuations of an elementary language L in a compatible relational system M , t is a term of L , and $\varphi(v) = \chi(v)$ for each variable v that occurs in t , then $\varphi(t) = \chi(t)$.
 - a. *Proof.* By recursion on the length of t .
 - b. *Corollary.* If t is a constant term, then all valuations give it the same value in M .
5. That value is called the *value* of the constant term in M .
6. *Theorem.* If φ, χ are valuations of an elementary language L in a compatible relational system and f is a formula of L , and $\varphi(v) = \chi(v)$ for each variable v that occurs free in f , then $\varphi(f) = \chi(f)$.
 - a. *Proof.* By recursion on the length of f .
 - b. *Corollary.* If f is a sentence (formula in which no variable occurs free) of an elementary language L , and M is a compatible relational system, then either $\varphi(f)$ is true for all valuations φ in M or it is false for all such φ .
7. We call the sentence *true* or *false* in M , accordingly. A formula f that is not a sentence is called *true* in M if $\varphi(f)$ is true for all valuations in M . (I'm not sure whether there's common agreement when to call f false.)
 - a. We call M a *model* of a set of formulas if all its members are true in M .
 - b. A sentence g is called a *semantic consequence* of a set S of sentences if it is true in every model of S .
 - c. The *semantic closure* of S is the set $c'(S)$ of all its semantic consequences.
8. *Theorem.* In fact, c' is a closure operator on the set of all formulas of any given elementary language. *Proof:* same as the proof of the corresponding theorem for Boolean logic.
9. *Theorem.* If the logical axioms (yet to be specified) of an elementary language L are true in every relational system M compatible with L , then for every set S of formulas of L , the syntactic closure $c(S)$ is a subset of the semantic closure $c'(S)$. *Proof.* If φ is a valuation of L in M , f and g are formulas of L , and $\varphi(f)$ and $\varphi(f \Rightarrow g)$ are true, then so is $\varphi(g)$, by definition of $\varphi(f \Rightarrow g)$.
10. *Some history.* It remains to specify the axioms required for the proof of the Gödel completeness theorem: $c(S) = c'(S)$.
 - a. This painfully detailed approach to its statement has amounted essentially to the use of set theory to formulate syntactic and semantic considerations that are commonly handled informally. Such a framework was not yet in use when Gödel developed a proof; his methods were only partially formal, and his framework not quite as general. But this degree of generality and precision is required for application of the completeness theorem in further logical and mathematical studies. Unfortunately, complete precision becomes so tedious that it tends to obscure the ideas. The present discussion, a compromise, has tried to avoid that. If you're used to describing computer languages and their properties it should seem ordinary. In the 1930s it was not.
 - b. The first paper that showed how to do this was Tarski's *Der Wahrheitsbegriff in den formalisierten Sprachen*. He wrote it first in Polish in 1931, then in German in 1936. An excellent English translation by the biologist J. H. Woodger is in Tarski [1956] 1983. You should find it quite readable. Tarski

spent many of its 126 pages making these definitions completely precise, long before anyone adapted such techniques to the design of computer languages. To the general public it looks like gobbledygook, and it became a bit notorious. When Tarski died, Herb Caen reported that Tarski was the logician who achieved fame by defining truth—so that the sentence “it is snowing” is true if and only if it is snowing! The gossip columnist and humorist of the *San Francisco Chronicle*, Caen kept this city laughing for fifty years.

- c. This approach can be modified rather easily to accommodate a larger set of truth values, using arbitrary matrices for the semantics of negation and implication, and lattice theoretic techniques for handling equality and quantification. But I won't do that.