

1. I've placed a lot of material on the website, the remainder of what I intend to generate for this course. (I will be editing some details, though.) We won't cover a great deal of it, but it's there and will remain so for your information. Here is a roadmap.
 - a. We're now finishing the material on the implicational part of Boolean logic. It's in previous outlines.
 - b. Today's lecture and this outline adapt that treatment to the full Boolean logic, with negation.
 - c. The next lecture and outline 38 will introduce first-order logic. I've finished preparing that material, through the Gödel completeness theorem. It's all contained in that outline today, but I'll continue the practice of editing and truncating an outline after each lecture to reflect what was covered, and moving the remaining material to the next outline.
 - d. Those lectures will introduce the notion of a first-order language, and the logical axioms required for the completeness theorem. I plan to describe the general structure of its proof, but will certainly skip over many technical details covered in the notes. The completeness theorem shows that for theories expressed in such a language, results obtained by ordinary mathematical reasoning can also be deduced according to the rules of first-order logic. That's why first-order logic is used to formalize so many mathematical theories.
 - e. The new unit *Zermelo–Fraenkel Set Theory*, now posted on the website, shows how to formalize in first-order logic the set theory used in earlier units. It specifies the language and axioms precisely, then briefly shows how the techniques developed in those units fit into the new framework.
 - f. Those earlier units did not give enough detail about the incorporation of natural number arithmetic within set theory to explain adequately how that process fits into the first-order framework. I've provided two units to address that.
 - g. The first, *Virtual Classes*, briefly describes a technique we all use habitually to appear to be reasoning with sets even though they're not essential to the discussion. Your explicit familiarity with that technique will allow me to limit the types of set-theoretic manipulations actually necessary for developing the natural number system.
 - h. Those few techniques are discussed in the unit *Natural Numbers*. There, I indulged a specific bias. To develop the natural number system, most set-theory texts use an axiom of infinity to guarantee existence of some infinite set, then use that to build the set of natural numbers. You learned a lot about natural numbers long before you ever heard of an infinite set; is that potent philosophical assumption really necessary for such elementary things? I should hope not. This unit shows how far you can go without making that

- assumption, and provides at least two way for doing so, within the first-order ZF framework.
- i. The *Natural Numbers* unit finishes by deriving the Peano axioms for natural-number arithmetic. Developing it in detail from that foundation, then integer, rational, real, and complex arithmetic is material for a whole course, only sporadically covered in this university. You're expected merely to believe that it can be done, without really seeing how in any course.
 - j. That should convince you that ZF set theory, as formalized in first-order logic, can handle the reasoning in this course, particularly the reasoning about languages that underlies its treatment of Boolean and first-order logic.
 - k. The final unit, *Gödel's Incompleteness and Tarski's Undefinability Theorems*, shows how to derive those results about the formalized ZF theory. They actually apply to any theory that can handle the linguistic concepts just mentioned. Since ZF is a very strong theory, it should not be surprising that it can do that. But those theorems were originally formulated to apply to a first-order theory of natural-number arithmetic, not a set theory. That original theory is much weaker than ZF, and much detailed work was required to show that it can handle the linguistic concepts. The treatment in this unit avoids that detail and concentrates on the logical structure of the Gödel and Tarski theorems.
2. Now extend the treatment of implicational propositional logic to the full Boolean.
 - a. To the alphabet A adjoin a second special symbol for denoting negation.
 - b. The string whose single member is an occurrence of the negation symbol is denoted here by $\neg$.
 - c. Define recursively a new set $F \subseteq A^*$ of *Boolean formulas*—use the same letter F so that you can reuse the previous text.
 - i. All variables are Boolean formulas.
 - ii. If p, q are Boolean formulas, then $\Rightarrow pq$ and $\neg p$ are Boolean formulas.
 - d. The definitions and discussion of syntactic consequence, deduction, and syntactic closure operator remain unchanged.
 - e. I'll adjoin some new forms of axioms to the set P . These will seem *ad hoc*, so I'll explain them only later, when I need to.
 - f. The deduction theorem and its proof remain unchanged.
 - g. The notion of implication matrix must be adapted: a *Boolean matrix* is a quadruple $\langle \mu, \nu, \Phi, T \rangle$ such that $\langle \mu, \Phi, T \rangle$ is an implication matrix and $\nu : \Phi \rightarrow \Phi$. The only Boolean matrix I'll consider is the material one, with $\nu(0) = 1$ and $\nu(1) = 0$. (But logicians have studied multivalued Boolean matrices deeply.)
 - h. The notion of a *valuation* $\varphi : V \rightarrow \Phi$ remains the same, but an additional recursion rule must be adopted for extending its domain to the set of all formulas: $\varphi(\neg p) = \nu(\varphi(p))$.
 - i. The definitions and discussion of semantic consequence, semantic closure operator, and tautology remain the same, with the proviso that when I specify any new axiom, I must check that it's a tautology.

- j. The definition of *theory* and the statement and proof of Lindenbaum's theorem remain the same.
- k. The statement of the completeness theorem remains the same.
- 3. But its proof must be adapted.
 - a. Part (e) of the proof contains these statements:
Thus, I may assume $f = (p \Rightarrow q)$ for some shorter formulas p, q . By the definition of μ_B , it is required to prove $(\varphi(p) = 1 \ \& \ \varphi(q) = 0) \Leftrightarrow f \notin M$.
 - b. They must be modified to read as follows:
Thus, I may assume $f = (p \Rightarrow q)$ **or $\neg p$** for some shorter formulas p, q . By the definitions of μ_B **and ν** , it is required to prove $(\varphi(p) = 1 \ \& \ \varphi(q) = 0) \Leftrightarrow f \notin M$ **or $\varphi(p) = 1 \Leftrightarrow f \notin M$ in those cases, respectively.**
 - c. Thus, it's necessary to prove the following statement: if $f = \neg p$ then $\varphi(p) = 1 \Leftrightarrow f \notin M$.
 - d. Just as the proofs of the other parts of the displayed statements required use of specific axioms, so does this new proof. Since it involves the negation symbol, it's reasonable to suppose the new proof requires a new axiom. Actually, *two* new axioms are required, at least temporarily, because the biconditional consists of two implications.
 - e. First, suppose that $f = \neg p$ and $\varphi(p) = 1$. It is required to show that $f \notin M$. Because p is shorter than f , the recursion hypothesis implies $p \in M$. If $f \in M$, then $\neg p \in M$. To handle that situation, I adopt as new axioms the tautologies $r \Rightarrow (\neg r \Rightarrow s)$ for all formulas r, s . In particular, $p \Rightarrow (\neg p \Rightarrow x)$ is now an axiom, and that would imply $x \in M$, contrary to the construction of M .
 - f. Conversely, suppose that $f = \neg p$ and $f \notin M$. It is required to show that $\varphi(p) = 1$. By the maximality of M , $x \in c(M \cup \{f\})$; by the deduction theorem, $(\neg p \Rightarrow x) = (f \Rightarrow x) \in M$. If also $p \notin M$, then for the same reason, $(p \Rightarrow x) \in M$. To handle that situation, I adopt as new axioms the tautologies $(r \Rightarrow s) \Rightarrow ((\neg r \Rightarrow s) \Rightarrow s)$ for all formulas r, s : thus $(p \Rightarrow x) \Rightarrow ((\neg p \Rightarrow x) \Rightarrow x)$ is now an axiom. That would imply $x \in M$, contrary to the construction of M .
- 4. The statement and proof of the compactness theorem remain the same. So, essentially do those of the decidability theorem. But it's convenient to state the full Boolean version here: there's an algorithm for deciding whether a Boolean formula can be deduced by *modus ponens* from the axioms
 - a. $p \Rightarrow (q \Rightarrow p)$,
 - b. $(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r))$,
 - c. $((p \Rightarrow q) \Rightarrow p) \Rightarrow p$,
 - d. $p \Rightarrow (\neg p \Rightarrow q)$, and
 - e. $(p \Rightarrow q) \Rightarrow ((\neg p \Rightarrow q) \Rightarrow q)$,
 where p, q, r may be any Boolean formulas, and no additional premisses.
- 5. This list of five forms of Boolean axioms is overlong. Church 1956, chapters 1–2, shows that axioms (c–e) can be replaced by a single more familiar form: the contraposition laws $(\neg q \Rightarrow \neg p) \Rightarrow (p \Rightarrow q)$ for all Boolean formulas p, q . That

requires explicit deductions of axioms (c–e) from axioms (a,b) and the contraposition laws. Those deductions are entertaining, but very involved and unlike arguments used in any other area of mathematics. Since they are unnecessary for this presentation, I won't describe them here.

6. This presentation of Boolean logic seems to have ignored all the connectives except $\Rightarrow$ and $\neg$. But not really.
 - a. For any formulas p, q define $(p \vee q) = ((p \Rightarrow q) \Rightarrow q)$. If φ is a valuation with respect to the material implication matrix, then $\varphi(p \vee q) = 1$ if and only if $\varphi(p) = 1$ or $\varphi(q) = 1$, as appropriate.
 - b. It's entertaining to investigate which of the familiar laws about the connectives $\Rightarrow$ and $\vee$ can be interpreted as deducible in the positive implicational propositional calculus.
 - c. For any formulas p, q we could define $(p \& q) = \neg(\neg p \vee \neg q)$ or $(p \& q) = \neg(p \Rightarrow \neg q)$. Other connectives can then be defined via their corresponding disjunctive normal forms.