

1. In Berkeley, Wednesday 30 April, 1900–2100, there is a free showing of Csicsery's new documentary about Julia Robinson, followed by a panel discussion by Bjorn Poonen, Martin Davis, and Dana Scott. Click this link for details:

http://www.msri.org/calendar/specialevents/SpecialEventInfo/353/show_specialevent

A professor at Berkeley, Julia was instrumental in solving Hilbert's tenth problem: no algorithm exists to decide whether any given Diophantine equation has an integer solution. The wife of professor Raphael Robinson, also noted for results in logic, she overcame severe gender bias to become the first female president of the American Mathematical Society, and earned several other major honors. Csicsery is a member of the Bay Area film-making community, noted for an earlier film about Paul Erdős. The film is to a major extent based on Reid 1996, a biography by Julia's sister Constance Reid. A San Francisco resident, Reid is also noted for earlier biographies of the mathematicians David Hilbert, Richard Courant, Jerzy Neyman, and Eric Temple Bell. Martin Davis and Dana Scott are major figures in modern logic, responsible for some of the results class members are studying for term papers.

2. Now, enlarge the axiom set P by including $((p \Rightarrow q) \Rightarrow p) \Rightarrow p$ for all formulas p, q : all instances of Peirce's law. I'll show that with this addition, the notions of syntactic and semantic consequence will coincide: $c'(S) = c(S)$ in general. In the generality of this presentation, proving that requires use of a maximal principle.
3. A set closed under the syntactic closure operator c is called a *theory*. According to results in the *Complete Lattices* unit, the theories, partially ordered by inclusion, constitute a complete lattice that contains $c(S)$ for any subset $S \subseteq F$.
4. *Theorem.* The union of a chain $\mathcal{C}$ of theories is a theory. *Proof.* It is required to show that $c(\bigcup \mathcal{C}) \subseteq \bigcup \mathcal{C}$. Let $q \in c(\bigcup \mathcal{C})$. It is required to show that $q \in \bigcup \mathcal{C}$. Since the syntactic closure operator is finitary, $q \in c(E)$ for some finite $E \subseteq \bigcup \mathcal{C}$. Since $\mathcal{C}$ is a chain, $E \subseteq C$ for some $C \in \mathcal{C}$. Thus $q \in c(C)$. Since C is a theory, $c(C) = C$. Thus $q \in C \subseteq \bigcup \mathcal{C}$.
5. *Lindenbaum's Theorem.* For any theory C and any formula q , if the family $\mathcal{U}$ of all theories that include C but do not contain q is nonempty, then $\mathcal{U}$ has a maximal member. *Proof.* The union of any chain in $\mathcal{U}$ is a theory by the previous theorem; obviously, it must include C and cannot contain q . By the Kuratowski–Zorn maximal principle, it must have a maximal member.
6. It is worth noting here that if the alphabet A is countable, then so is the set of strings A^* , hence also the set F of formulas, and thus the version of the maximal principle needed in that case does not depend on the axiom of choice.
7. *Completeness Theorem.* With the new axiom set and the material implication matrix, the notions of syntactic and semantic consequences coincide: $S \vdash q$ if and only if $S \models q$, and $c(S) = c'(S)$ for any set S of formulas and any formula q .

Proof. It is required to prove that with the new axiom set, $c'(S) \subseteq c(S)$. Suppose x is a formula but $x \notin c(S)$. It is required to prove that $x \notin c'(S)$.

- a. By Lindenbaum's theorem, there is a theory $M \subseteq F$ that includes $c(S)$ but does not contain x , such that $x \in c(M \cup \{p\})$ for any formula $p \in M$.
 - b. Consider the material implication matrix $\langle \mu_B, \{0, 1\}, \{1\} \rangle$ and the valuation $\varphi : V \rightarrow \{0, 1\}$ defined by setting $\varphi(p) = 1 \Leftrightarrow p \in M$, for each variable $p \in V$.
 - c. Extend the domain of φ to the set F of all formulas by applying the rule $\varphi(p \Rightarrow q) = \varphi(p) \mu_B \varphi(q)$ recursively.
 - d. I shall show that the resulting function $\varphi : F \rightarrow \{0, 1\}$ is the characteristic function of M . It then follows that $\varphi(p) = 1$ for every formula $p \in S$, but $\varphi(q) = 0$, hence $q \notin c'(S)$.
 - e. I shall show that $\varphi(f) = 0 \Leftrightarrow f \notin M$ for every formula $f \in F$, by recursion on the length of f . Assume this equivalence is true for all formulas shorter than f . If f is a variable, the equivalence is true by definition of φ . Thus, I may assume $f = (p \Rightarrow q)$ for some shorter formulas p, q . By the definition of μ_B , it is required to prove $(\varphi(p) = 1 \ \& \ \varphi(q) = 0) \Leftrightarrow f \notin M$.
 - i. If $\varphi(p) = 1 \ \& \ \varphi(q) = 0$ then $p \in M$ and $q \notin M$, because p and q are shorter than f . That entails $f \notin M$ —that is, $(p \Rightarrow q) \notin M$ —because q is a syntactic consequence of p and $p \Rightarrow q$.
 - ii. Next suppose $\varphi(q) \neq 0$, so that $q \in M$ because q is shorter than f . It is required to show that $f \in M$. This follows because f —that is, $(p \Rightarrow q)$ —is a syntactic consequence of q and the axiom $q \Rightarrow (p \Rightarrow q)$.
 - iii. Finally, suppose $\varphi(p) \neq 1$.
 - (1) It must be shown that $f \in M$. To this end I will suppose $f \notin M$, and derive a contraction.
 - (2) $p \notin M$ because $\varphi(p) \neq 1$ and p is shorter than f .
 - (3) $x \in c(M \cup \{p\})$ by step (a).
 - (4) $(p \Rightarrow x) \in c(M) = M$ by (3) and the deduction theorem.
 - (5) $x \in c(M \cup \{f\}) = c(M \cup \{p \Rightarrow q\})$ by (1) and step (a) again.
 - (6) $p \in c(M \cup \{(p \Rightarrow q), (x \Rightarrow p)\})$ by (5) because p is a consequence of x and $(x \Rightarrow p)$.
 - (7) $((p \Rightarrow q) \Rightarrow p) \in c(M \cup \{x \Rightarrow p\})$ by (6) and the deduction theorem.
 - (8) $x \in c(M \cup \{x \Rightarrow p\})$ by (4, 7) since x is a consequence of a Peirce axiom, $((p \Rightarrow q) \Rightarrow p)$, and $(p \Rightarrow x)$.
 - (9) $((x \Rightarrow p) \Rightarrow x) \in c(M) = M$ by (8) and the deduction theorem.
 - (10) $x \in M$ by (9) since x is a consequence of $((x \Rightarrow p) \Rightarrow x)$ and a Peirce axiom. But this contradicts the initial assumption about x .
8. The use of the Peirce axioms in the previous outline and in the proof of the completeness theorem are prime examples of *axiom-pushing*: obtaining simply stated, powerful results through complicated and clever formulation and application of simple axioms.
 9. *Compactness Theorem.* The semantic closure operator for the material implication matrix is finitary.
 - a. *Proof.* Suppose $q \in c'(S)$ for some subset $S \subseteq F$. If c is the syntactic closure operator for the material implication matrix, then $c'(S) = c(S)$ by the com-

pleteness theorem. Since c is finitary, $q \in c(E) = c'(E)$ for some finite $E \subseteq S$.

- b. Results like the compactness theorem have many applications. Proving them without such a detour through dirty proof theory would be quite hard.
- 10. *Decidability Theorem*. There is an algorithm for deciding whether an implicational formula can be deduced by *modus ponens* from the axioms
 - a. $p \Rightarrow (q \Rightarrow p)$,
 - b. $(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r))$, and
 - c. $((p \Rightarrow q) \Rightarrow p) \Rightarrow p$,

where p, q, r may be any implicational formulas, and no additional premisses.

Proof. There is an obvious algorithm for deciding whether an implicational formula is a tautology with respect to the material implication matrix.

- 11. *Historical notes*
 - a. The syntactical consequences of axioms (10a) and (10b)—that is, $c(\phi)$ with that axiom set—form the theory known as the *positive implicational propositional calculus*.
 - b. The consequences of all the axioms (10)—that is, the tautologies with respect to the material implication matrix—form the theory known as the *implicational propositional calculus*.
 - c. The best source for historical information and the lay of the land in this area of logic, up to fifty years ago, is Church 1956, chapters 1–2. I’m discovering that I don’t know enough about the history since then to report it here. I learned most of this material from Church 1956, but do not use his approach.
 - d. The general flavor of the present approach reflects that of several papers by Tarski in the 1920s and 1930s, reprinted in [1956] 1983.
 - e. The use of a three-valued matrix μ_B is an example of the theory of *many-valued* logic, which is quite extensive.
 - f. The material covered here immediately suggests some questions.
 - i. Can axiom (a) be deduced from (b) or from (b) and (c)? Can axiom (b) be deduced from (a) or from (a) and (c)? The answers to both are *no*; the proofs use 3- and 4-valued implicational matrices.
 - ii. Can the positive implicational propositional calculus be described by means of a many-valued matrix? I don’t know.
 - iii. Is there an algorithm for determining whether an implicational formula is a consequence of axioms (10a) and (10b) alone? The answer is *yes*. The techniques used for that algorithm come from the theory of natural deduction, an alternative approach to deducibility unfamiliar to me. But I don’t think the result is very deep.
 - g. Another approach to this subject uses only one axiom of each form, but an additional proof rule that permits substitution of formulas for variables within a deduction. The results are similar, except that in one critical place in decidability theory it is useful to have only finitely many axioms. The benefit of that approach is obtained with a cost: awful notation in describing deductions.
 - h. There are many other fascinating partial systems of Boolean logic.