

1. Next, for an arbitrary matrix, I'll define semantic notions parallel to syntactic consequence and closure.
 - a. A formula $q \in F$ is called a *semantic consequence* of a set $S \subseteq F$ of *assumptions*—written $S \models q$ —if for every valuation φ related to the matrix, $\varphi[S] \subseteq T \Rightarrow \varphi(q) \in T$.
 - b. That is, the semantic consequences of a set of assumptions are those formulas that have true values whenever all the assumptions do.
 - c. The *semantic closure* of a subset $S \subseteq F$ is the set $c'(S)$ of all its semantic consequences.
2. *Theorem.* In fact, c' is a closure operator on F in the sense of the *Complete Lattices* unit. *Proof.*
 - a. $S \subseteq c'(S)$ for any $S \subseteq F$, obviously.
 - b. Therefore $c'(S) \subseteq c'(c'(S))$. Next, I'll prove the reverse inclusion, to get equality here. Suppose $q \in c'(c'(S))$. To show that $q \in c'(S)$, assume that φ is a valuation for which $\varphi[S] \subseteq T$; I must show that $\varphi(q) \in T$. The assumption in the previous sentence implies that $\varphi(r) \in T$ for each $r \in c'(S)$. According to the first sentence in (i), $\varphi(q) \in T$ as desired.
 - c. If $R \subseteq S \subseteq F$, then $c'(R) \subseteq c'(S)$, obviously.
3. The remainder of this discussion is devoted to studying the relationship between syntactic and semantic consequences and closure.
4. The *tautologies* determined by an implication matrix are the formulas p such that $\varphi(p) \in T$ for every valuation φ related to the matrix.
5. *Theorem.* If all the axioms in P are tautologies with respect to a given implication matrix, then all syntactic consequences of a subset $S \subseteq F$ are also semantic consequences. That is, $c(S) \subseteq c'(S)$ *Proof.* Suppose $S \vdash q$. Then there is a deduction of q from S : a sequence of formulas $p_0, \dots, p_{n-1}$ such that $p_{n-1} = q$ and for each $m < n$,
 - a. $p_m \in S$ —in this case $p_m \in c'(S)$ by the previous theorem—or
 - b. $p_m \in P$ —in this case $p_m \in c'(S)$ by hypothesis—or
 - c. there exist $k, l < m$ such that $p_k = (p_l \Rightarrow p_m)$.
 I'll show by recursion that $p_m \in c'(S)$ for each $m < n$. The recursion hypothesis is that $p_j \in c'(S)$ for each $j < m$. To show that $p_m \in c'(S)$ in case (c), suppose that φ is a valuation for which $\varphi[S] \subseteq T$. Then $\varphi(p_l \Rightarrow p_m), \varphi(p_l) \in T$, hence $\varphi(p_m) \in T$ by (12c).
6. All the axioms adopted so far are familiar tautologies with respect to the material implication matrix. Therefore, all syntactic consequences of a subset $S \subseteq F$ are also semantic consequences with respect to that matrix. That should not be surprising, since much of the apparatus constructed so far is merely a set-theoretic implementation of familiar informal Boolean theory.
7. Now I'll introduce a different matrix, to provide a counterexample to the converse of the previous statement.

- a. Consider the matrix $\langle \mu_B, \Phi, T \rangle$ with

$\Phi = \{0, \frac{1}{2}, 1\}$	$T = \{1\}$	p	q	$p \mu_B q$
		1	1	1
Be sure to check		1	$\frac{1}{2}$	$\frac{1}{2}$
$(\forall p, q \in \Phi)[p \in T$		1	0	0
$\& p \mu_B q) \in T \Rightarrow q \in T]!$		$\frac{1}{2}$	1	1
		$\frac{1}{2}$	$\frac{1}{2}$	1
		$\frac{1}{2}$	0	0
		0	1	1
		0	$\frac{1}{2}$	1
		0	0	1

- b. It's tedious but not hard to check that all the axioms adopted so far are tautologies with respect to μ_B .
- c. I could give a very informal argument why that shouldn't be surprising, but it wouldn't be very convincing, so I won't.
- d. Therefore, all syntactic consequences of a subset $S \subseteq F$ are also semantic consequences with respect to that matrix.
- e. It's easy to check that if p, q are variables, then $((p \Rightarrow q) \Rightarrow p) \Rightarrow p$, often called *Peirce's law*, is *not* a tautology with respect to μ_B . Thus Peirce's law is *not* a syntactic consequence of ϕ , with the current axiom set.
- f. That shouldn't be surprising, because both Peirce's law and μ_B seem far-fetched.
- g. What is surprising to me is that Peirce's law *is* a tautology with respect to the material implication matrix: check it! Thus Peirce's law *is* a semantic consequence of ϕ under material implication.
- h. Thus, with the current axiom set, $c'(S) \subsetneq c(S)$ in general.
- i. This method of showing unprovability, and I think this specific result, was introduced by Paul Bernays in 1919, but published only in 1926.