

1. *Homework list.* I'd inadvertently left unlisted in Outline 31 one problem on fell sweeping. I've fixed that.
2. *Deduction Theorem.* If $S \subseteq F$ and $q, r \in F$, then $S \cup \{q\} \vdash r$ if and only if $S \vdash (q \Rightarrow r)$. *Proof.*
 - a. If $S \vdash (q \Rightarrow r)$ then you can deduce r from $S \cup \{q\}$ by appending q to a deduction of $q \Rightarrow r$ from S .
 - b. Now prove the converse. Suppose $S \cup \{q\} \vdash r$.
 - i. There exists a positive $n \in \mathbb{N}$ and formulas $p_0, \dots, p_{n-1}$ such that $p_{n-1} = r$ and for each $m < n$,
 - (1) $p_m \in S$, or
 - (2) $p_m \in P$, or
 - (3) there exist $k, l < m$ such that $p_k = (p_l \Rightarrow p_m)$, or
 - (4) $p_m = q$.
 - ii. From this deduction of r from $S \cup \{q\}$ construct a new sequence of formulas $q \Rightarrow p_m$ —thus, $q \Rightarrow p_{n-1}$ is the same as $q \Rightarrow r$.
 - iii. Before each formula $q \Rightarrow p_m$ insert into the new sequence one, zero, two, or four formulas according to the following scheme.
 - (1) If $p_m \in S$, insert the axiom $p_m \Rightarrow (q \Rightarrow p_m)$.
 - (2) If $p_m \in P$, don't insert anything.
 - (3) If there exist $k, l < m$ such that $p_k = (p_l \Rightarrow p_m)$, insert
 - (a) the axiom $(q \Rightarrow (p_l \Rightarrow p_m)) \Rightarrow ((q \Rightarrow p_l) \Rightarrow (q \Rightarrow p_m))$ and
 - (b) the formula $(q \Rightarrow p_l) \Rightarrow (q \Rightarrow p_m)$, which follows from (a) and $q \Rightarrow (p_l \Rightarrow p_m)$, which is the same as $q \Rightarrow p_k$.
 - (4) If $p_m = q$, insert formulas 8a–8d from the previous outline.
 - iv. The resulting new sequence is a deduction of $q \Rightarrow r$ from S .
 - (1) Notice that $q \Rightarrow p_m$ follows from iii.3b and $q \Rightarrow p_l$, and
 - (2) the formulas inserted before $q \Rightarrow p_m$ in iii.4, together with that formula, constitute a deduction of $q \Rightarrow q$ from ϕ .
3. The proofs of the previous theorems illustrate the field of logic known as *proof theory*: deep syntactical analysis of mathematical deductions. Clearly, the axiom set P was chosen precisely to make the proof of the deduction theorem work.
4. Now, I'll start introducing *semantical* ideas for this fragment of Boolean logic: a primitive analysis of possible *meanings* of formulas. Again, I'll make as few commitments as possible so that the same framework will work in several contexts.
5. I'll use the term *implication matrix* for a triple $\langle \mu, \Phi, T \rangle$ such that
 - a. $\mu : \Phi \times \Phi \rightarrow \Phi$,
 - b. $T \subseteq \Phi$, and
 - c. $(\forall p, q \in \Phi)[p \in T \ \& \ p \mu q \in T \Rightarrow q \in T]$.

There's no standard term for this notion, but this one has been used often in the past; it stems from the Latin *mater*, for *mother*. Elements of Φ are called *truth*

values; those in Φ are called *true*. Requirement (c) should suggest the *modus ponens* rule.

6. The *material* implication matrix is the triple $\langle \mu, \Phi, T \rangle$ with

$\Phi = \{0, 1\}$	$T = \{1\}$	p	q	$p \mu q$
		1	1	1
		1	0	0
		0	1	1
		0	1	1

This should look familiar. Be sure to check requirement (c). You should think of material implication as the typical example, but we'll also use another one later.

7. Related to any implication matrix $\langle \mu, \Phi, T \rangle$ are *valuations*: functions φ from the set V of variables to the set Φ of truth values.
- A valuation's domain can be extended to the set of all formulas by applying the rule $\varphi(p \Rightarrow q) = \varphi(p) \mu \varphi(q)$ recursively.
 - This rather complicated framework is merely the set-theoretic implementation of the notion of a single line in a conventional truth table, generalized to permit an arbitrary set of truth values and an arbitrary interpretation of the implication symbol that is compatible with *modus ponens*.