

1. I'm going to implement in this set-theoretic syntactic context a very limited notion of proof. It's based on a deduction rule that will be familiar, but limitation to that rule alone may seem arbitrary. It's based on a set P of axioms that I'll leave unspecified for a while. These will seem entirely arbitrary.
2. A formula $q \in F$ is called a *syntactic consequence* of a set $S \subseteq F$ of *premisses*—written $S \vdash q$ —if there exists a positive $n \in \mathbb{N}$ and formulas $p_0, \dots, p_{n-1}$ such that $p_{n-1} = q$ and for each $m < n$,
 - a. $p_m \in S$, or
 - b. $p_m \in P$, or
 - c. there exist $k, l < m$ such that $p_k = (p_l \Rightarrow p_m)$.
 Such a sequence $p_0, \dots, p_{n-1}$ of formulas is called a *deduction* of q from S .
3. In those cases we say that
 - a. p_m is a premiss, or
 - b. p_m is an axiom, or
 - c. p_m follows from p_k, p_l by *modus ponens*.
 The latter term stems from the use of this deduction rule among others in ancient logic. The symbol $\vdash$ is called a *turnstile*. We often write $\vdash q$ to mean $\phi \vdash q$.
4. The *syntactic closure* of a set $S \subseteq F$ is $c(S) = \{q \in F : S \vdash q\}$. Warning: this notion, like that of deduction, depends on the details of the axiom set P .
5. *Theorem.* In fact, c is a closure operator on F in the sense of the *Complete Lattices* unit. *Proof.*
 - a. $S \subseteq c(S)$ for any $S \subseteq F$, by (2a).
 - b. Therefore $c(S) \subseteq c(c(S))$. Next, I'll prove the reverse inclusion, to get equality here.
 - i. Suppose $q \in c(c(S))$. Then there is a deduction $p_0, \dots, p_{n-1}$ of $p_{n-1} = q$ from $c(S)$: for each $m < n$,
 - (1) $p_m \in c(S)$, or
 - (2) $p_m \in P$, or
 - (3) there exist $k, l < m$ such that $p_k = (p_l \Rightarrow p_m)$.
 - ii. From this sequence of formulas construct a new one as follows. For each $m < n$ such that (1) holds but not (2, 3), insert all but the last formula of a deduction of p_m from S .
 - iii. The new sequence is a deduction of a from S , so $q \in c(S)$, as desired.
 - c. If $R \subseteq S \subseteq F$, then $c(R) \subseteq c(S)$ because every deduction from R is a deduction from S .
6. *Theorem.* The syntactic closure operator is *finitary*: if $q \in c(S)$ for any $S \subseteq F$, then $q \in c(E)$ for some finite $E \subseteq S$.
 - a. *Proof.* Let E consist of the formulas in S that occur in a deduction of q from S .
7. The first set P of axioms to study is the set of all formulas
 - a. $p \Rightarrow (q \Rightarrow p)$

$$\text{b. } (p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r))$$

where $p, q, r \in F$. Thus there are infinitely many axioms, of these two forms.

8. *Theorem.* $\vdash (q \Rightarrow q)$ for any $q \in F$. *Proof.*

$$\text{a. } (q \Rightarrow ((q \Rightarrow q) \Rightarrow q)) \Rightarrow ((q \Rightarrow (q \Rightarrow q)) \Rightarrow (q \Rightarrow q))$$

axiom 11b

$$\text{b. } (q \Rightarrow ((q \Rightarrow q) \Rightarrow q))$$

axiom 11a

$$\text{c. } (q \Rightarrow (q \Rightarrow q)) \Rightarrow (q \Rightarrow q)$$

(a, b, *modus ponens*)

$$\text{d. } (q \Rightarrow (q \Rightarrow q))$$

axiom 11a

$$\text{e. } q \Rightarrow q$$

(c, d, *modus ponens*)