

1. *In class, the Cardinals II unit*

- a. From Mr. Krogh-Freeman's result that $\mathbb{R} \approx (0, 1)$ and lavish use of the Cantor–Bernstein theorem, it follows that all nontrivial intervals of real numbers are equinumerous. Cantor's 1878 proof of that result was more complicated than ours because that theorem hadn't been established yet.
- b. Mr. Miranda discussed *routine exercise 2*, on the cardinal α of the set A of irrational numbers. From $A \cap \mathbb{Q} = \emptyset$ and $A \cup \mathbb{Q} = \mathbb{R}$ it follows that $\alpha + \omega = 2^\omega$. Then $\alpha < \omega$ would imply $\omega = \alpha + \omega = 2^\omega$, contrary to Cantor's theorem; therefore, $\alpha \not< \omega$. By the comparison theorem, $\omega \leq \alpha$ and thus $\alpha = \alpha + \omega = 2^\omega$.
- c. Cantor knew that result, but could only conjecture the comparison theorem. How could he have computed α ? Clearly, $\alpha \leq 2^\omega$, from which the result would follow from $2^\omega \leq \alpha$ and the Cantor–Bernstein theorem, which he also conjectured, perhaps on stronger grounds. He could have proved that $2^\omega \leq \alpha$ by considering the injection $f: \mathbb{R} \rightarrow A$ such that for all $c \in \mathbb{N}$ and all sequences of digits d_n such that $(\forall m \in \mathbb{N})(\exists n \in \mathbb{N})[m < n \ \& \ d_n \neq 9]$,

$$f(c + .d_0d_1d_2d_3d_4\dots) = c + .d_00d_100d_2000d_30000d_4\dots$$

Finding out how he actually did it would be a good miniproject.

- d. Ms. Yaggie discussed *routine exercise 9*, using a related argument: she showed that $\mathbb{R} \times \mathbb{R} \preceq \mathbb{R}$ by considering the injection $j: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ such that for all sequences of digits d_n, e_n as described in the previous item and all finite sequences of digits $b_0, \dots, b_m$ and $c_0, \dots, c_m$,

$$\begin{aligned} j(<b_0\dots b_m.d_0d_1d_2d_3d_4\dots, c_0\dots c_m.e_0e_1e_2e_3e_4\dots>) \\ = b_0c_0b_1c_1\dots b_m c_m.d_0e_0d_1e_1\dots \end{aligned}$$

(You can represent any pair $b + d, c + e$ of numbers such that $b, c \in \mathbb{N}$ and $d, e \in (0, 1)$ this way by padding the decimal expansion of one of b, c on the left with zeroes if it should be shorter than the decimal expansion of the other.) From this result it follows that $2^\omega 2^\omega \leq 2^\omega$. But j is not surjective: its range does not contain $0.090909\dots$, for example. Thus to show $2^\omega 2^\omega = 2^\omega$, you need to observe that clearly $2^\omega \leq 2^\omega 2^\omega$, then apply the Cantor–Bernstein theorem.

- e. I think that since Cantor had no proof of the Cantor–Bernstein theorem, he used a similar “interleaving” argument with continued-fraction representations of real numbers, which is harder in structure, but doesn't suffer from the endless-nines ailment.
- f. Mr. Rizzolo discussed *routine exercise 10*. Since $2^\omega 2^\omega = 2^\omega$, there exists a bijection $h: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$. Given any function $f: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$, define $g: \mathbb{R} \rightarrow \mathbb{R}$ by setting $g = f \circ h^{-1}$. Then $f = g \circ h$. That is, every real function of two

variables results from composing some function of *one* variable with a *single* given function of two variables.

- g. Ms. Evans discussed *routine exercise 12*, on the cardinal α of the set A of all convergent sequences of real numbers. Since $A \subseteq \mathbb{R}^{\mathbb{N}}$, $\alpha \leq (2^{\omega})^{\omega} = 2^{\omega\omega} = 2^{\omega}$. To show that $\alpha = (2^{\omega})^{\omega} = 2^{\omega\omega} = 2^{\omega}$, consider the injection that maps each $x \in \mathbb{R}$ to the convergent sequence all of whose terms equal x .
 - h. A number of substantial problems in this unit, not assigned, involve concepts covered in real analysis courses. If you have some familiarity with those concepts, you can gain a much better grasp of them by considering those problems.
2. *In class: the comparability theorem*
- a. Early in his work, Cantor conjectured this case of the comparability theorem: $\alpha \leq \beta$ or $\beta \leq \alpha$ for any cardinals α, β . That disjunction is often stated “ $\alpha < \beta$ or $\alpha = \beta$ or $\alpha \geq \beta$ ” and called the *trichotomy* theorem. From that result it follows, without further use of the axiom of choice, that for every infinite set X there exists an injection from $\mathbb{N}$ to X . *Proof.*
 - i. Suppose X is an infinite set, but
 - ii. no such injection existed.
 - iii. Then $\omega \not\leq \#X$, and by the trichotomy theorem, $\#X \leq \omega$ —that is, there would be an injection $h : X \rightarrow \mathbb{N}$.
 - iv. And h would have a bijective left inverse $e : \text{Rng } h \rightarrow X$.
 - v. Suppose $\text{Rng } h$ were unbounded: $(\forall n \in \mathbb{N})(\exists x \in X)[n < h(x)]$.
 - (1) By recursion techniques employed in earlier outlines, I could define an injection $f : \mathbb{N} \rightarrow \text{Rng } h$. (This uses the natural well-ordering of $\text{Rng } h$.)
 - (2) Then $e \circ f : \mathbb{N} \rightarrow X$ injectively, contrary to (ii).
 - vi. Thus $\text{Rng } h$ would have to be bounded:

$$(\exists n \in \mathbb{N})[\text{Rng } h \subseteq \{l \in \mathbb{N} : l < n\}].$$
 - (1) By recursion, I could find $m \in \mathbb{N}$ and define a bijection $g : \text{Rng } h \rightarrow \{l \in \mathbb{N} : l < m\}$. (This uses the natural well-ordering of $\mathbb{N}$.)
 - (2) Then $g \circ h : X \rightarrow \{l \in \mathbb{N} : l < m\}$ bijectively, contrary to (i).
 - vii. All paths from (ii) led to contradiction, hence (ii) must be false, which was to be proved.
 - b. Discussion of this topic was confused until around 1903, because questions concerning the formulation of the axiom of choice, the definitions of finite and infinite, the well-ordering, the Cantor–Bernstein theorem, and the trichotomy theorem were so closely interrelated. It was known early that the well-ordering theorem implied trichotomy, which implied the equivalence of the two most common definitions of finiteness. Proof of the Cantor–Bernstein theorem came next. Then Zermelo showed in 1903 that the axiom implies well-orderability. (The converse is trivial, but wasn’t thought of until Zermelo stated the axiom appropriately.) Only in 1915 did F. Hartogs show that the trichotomy theorem implies the axiom of choice. That result is covered in

substantial problems 14, 15 of the *Cardinals II* unit, which were not assigned. Problem 15 continues, showing that Hartogs's result easily implies that, properly stated, the familiar infinite-cardinal-arithmetic theorems each imply the axiom of choice. Problems 14, 15 are arranged delicately to avoid the use of ordinal numbers, which Hartogs employed in his proof.