

1. *Assignment:*
 - a. Read the *Boolean Logic* unit and the part of Tarski[1946] 1995, chapter 2, sections 12–15. Hand in the following.
 - b. Tarski [1946] 1995, chapter 2:
 - i. Exercise 12 and the corresponding parts of exercise 13. “True” means “tautologous”, and “false” means “not tautologous”.
 - ii. Exercise 15b
 - c. *Boolean Logic*
 - i. Exercise 1
 - ii. Exercise 2, just one sentence. (Don’t everybody give me the same one!)
 - iii. Use the fell-swoop method to verify that the self-distributivity law and Peirce’s law are in fact tautologies.
 - iv. Exercise 4
 - v. Exercise 5: Don’t do the proof, just find the conjunctive normal form.
 - vi. Exercise 6.
 - d. I discussed this assignment briefly.
2. *In class, the Cardinals II unit*
 - a. Ms. Morgan discussed *routine exercise 4*. Given $f : \mathbf{R} \rightarrow \omega$, define an equivalence relation E on $\mathbf{R}$ by setting $x E y \Leftrightarrow f(x) = f(y)$, for all $x, y \in \mathbf{R}$. For each such x let $E_x = \{y \in \mathbf{R} : x E y\}$, so that $x \in E_x$. Let $I = \{E_x : x \in \mathbf{R}\}$. Each member of I is nonempty, so there is a function $c \in \mathbf{X}_{S \in I} S$. The members of I are disjoint, so $c : I \rightarrow \mathbf{R}$ injectively, and thus $\#I \leq \#\mathbf{R} = 2^\omega$. She claimed that the cardinal of some $S \in I$ must $= 2^\omega$. For otherwise, the Zermelo–König theorem (substantial problem 6 in the *Axiom of Choice* unit) would imply that $2^\omega = \mathbf{R} = \sum_{S \in I} \#S < \prod_{S \in I} 2^\omega = 2^\omega \#I \leq 2^\omega 2^\omega = 2^{\omega \omega} = 2^\omega$, which is impossible. Therefore, $\mathbf{R}$ has a subset of the same cardinality on which f is constant.
 - b. Mr. Krogh-Freeman discussed *routine exercise 8*. In fact, the function $\frac{1}{2} + (1/\pi) \arctan$ is a bijection from $\mathbf{R}$ to the open interval $(0, 1)$. We’ll use the theory of cardinality next week to show that all nontrivial intervals of real numbers are equinumerous. But except for the open intervals, the required bijections are not so easy to find.
 - c. Ms. Gross discussed *substantial problem 2*. Let X be an infinite set and α a nonzero cardinal $\leq \#X$. There must be a nonempty set A such that $\#A = \alpha$. Thus there must be a bijection $f : A \times X \rightarrow X$. For each $x \in X$ let $A_x = f[A \times \{x\}]$ and define $g_x : A \rightarrow A_x$ by setting $g_x(a) = \langle a, x \rangle$. Since g_x is bijective, $f \circ g_x : A \rightarrow A_x$ bijectively, and thus $\#A_x = \alpha$. Thus X is the disjoint union of the family of sets A_x , each of which has cardinal α , and the family itself has the same cardinal as X .

3. *History of the cardinal arithmetic theorem,*

$$\alpha \leq \beta \text{ \& } \beta \text{ infinite} \Rightarrow \alpha + \beta = \alpha\beta = \beta.$$

- a. Cantor proved and published a number of special cases of this theorem. Gerhard Hessenberg reported (1906, 595) that Cantor had proved, probably in the 1890s, but had not published, that the theorem holds for *alephs* α, β —cardinals of well-ordered sets. But I haven't found the general problem stated in the literature through 1900. I don't think Arthur Schoenflies mentioned it in his major 1900 survey.
- b. During the next several years discussions of this subject were mixed up with discussions of the Cantor–Bernstein theorem, the comparison theorem, and the validity of proofs that employed infinitely many arbitrary choices.
- c. [Click here](#) for a cartoon that illustrates this situation.
- d. In 1901 Zermelo studied the class of cardinals β for which the cardinal arithmetic theorem should hold. His methods, on page 38, yield the argument

$$2\beta = \beta \text{ \& } \alpha \leq \beta \Rightarrow \beta \leq \alpha + \beta \leq \beta + \beta = 2\beta = \beta \Rightarrow \alpha + \beta = \beta$$

for any α, β .

- e. In 1902 Alfred North Whitehead stated the cardinal arithmetic theorem as an open problem. He included Russell's proof that $2\beta = \beta$ follows from the hypothesis that β be the cardinal of the union of a disjoint family of countably infinite sets: for then $\beta = \omega\gamma$ for some γ , hence $2\beta = 2(\omega\gamma) = (2\omega)\gamma = \omega\gamma = \beta$. For selected original text, [click here](#). Russell's hypothesis is a special case of the result in substantial problem 2.
- f. In 1904a, 1904b, Jourdain claimed to have proved the cardinal arithmetic theorem for all infinite β . The first paper presented an informal argument that β should be an aleph, then another informal argument replacing Russell's unproved hypothesis and concluding with $2\beta = \beta$. The second paper included an involved recursive argument that concluded with $\beta^2 = \beta$. That equation would yield the rest of the cardinal arithmetic theorem:

$$\beta^2 = \beta \text{ \& } \alpha \leq \beta \Rightarrow \beta \leq \alpha\beta \leq \beta \cdot \beta = \beta^2 = \beta \Rightarrow \alpha\beta = \beta.$$
- g. Jourdain's arguments for β 's being an aleph, $2\beta = \beta$, and $\beta^2 = \beta$ hardly seem conclusive, and evidently were not taken too seriously, since he is not now credited for the proof of the cardinal arithmetic theorem.
- h. In [1904] 1970 Ernst Zermelo published the first convincing proof of the well-ordering theorem: every set can be well-ordered, hence every cardinal is an aleph. He claimed that should settle the cardinal arithmetic theorem for all infinite β . [Click here](#) for the original text. Hessenberg reported (1906, 595) that Zermelo had a proof of that, and would publish it soon. But it never appeared. As mentioned elsewhere, Zermelo's paper excited a huge controversy.
- i. [Click here](#) for a portrait of Zermelo.

- j. Hessenberg published in 1906, §20, the first complete proof of the cardinal arithmetic theorem for all infinite β . It consists in essentially these steps
 - i. selecting a well-ordered set B such that $\beta = \#B$;
 - ii. using recursion on B to construct an injection from $B \times B$ into B , which implies $\beta^2 = \beta$;
 - iii. deducing the theorem from the argument displayed earlier and $2\beta \leq \beta \cdot \beta = \beta^2 = \beta \Rightarrow 2\beta = \beta$.
- k. Hessenberg's method remained standard until 1944, when Max Zorn used the maximal principle to construct neat arguments that replace the ordinals and recursions used earlier.
- l. Click here for a portrait of [Zorn](#).
- m. The proof of $\beta^2 = \beta$ presented in these notes requires that the equation $2\beta = \beta$ be established first. Hessenberg's recursive proof did not.