

1. *In class, the Cardinals II unit*
 - a. I discussed the proof that $\omega + \omega = \omega$. In particular, it was easy to find a formula for the required bijection from $\mathbb{N}$ to $(\mathbb{N} \times \{0\}) \cup (\mathbb{N} \times \{1\})$.
 - b. For $n \in \mathbb{N}$, it follows that $\omega \leq n + \omega \leq \omega + \omega = \omega$, so $n + \omega = \omega$. This use of the weak antisymmetry of $\leq$ —i.e., the Cantor–Bernstein theorem—is typical. It lets us avoid nasty proofs.
 - c. Some presentations of cardinal arithmetic include subtraction. It’s not very interesting. Nevertheless, you can define $\alpha - \beta$ as the cardinal δ such that $\alpha = \beta + \delta$, provided you can show there is a unique such δ .
 - d. Thus if $n \in \mathbb{N}$ then $\omega - n = \omega$.
 - e. [Click here](#) for a cartoon. It calls to mind the band-bus chant that starts, “Ninety-nine bottles of beer on the wall, ninety-nine bottles of beer. A bottle of beer fell off the wall, ninety-eight bottles of beer on the wall....” Set theory transcend puerility.
 - f. If α is the cardinal of an infinite set A then $\omega + \alpha = \alpha$. *Proof.* We can assume A is disjoint from $\mathbb{N}$. Since $\omega \leq \alpha$, there is an injection $a : \mathbb{N} \rightarrow A$. Define $f : \mathbb{N} \cup A \rightarrow A$ by setting $f(n) = a_{2n}$ and $f(a_n) = a_{2n+1}$ for each $n \in \mathbb{N}$, and $f(x) = x$ for each $x \in A - \text{Rng } a$.
 - g. I discussed the proof that $\omega\omega = \omega$. It’s visually obvious that there is a bijection from $\mathbb{N}$ to $\mathbb{N} \times \mathbb{N}$, but it would be very hard to write down a formula for the obvious zig-zag function.
2. Martin Davis (1958, 43–45) proved that the formula

$$f(\langle x, y \rangle) = \frac{1}{2}((x + y)^2 + 3x + y)$$
 defines a bijection from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$ and gave formulas for the components of its inverse. I don’t remember whether that has any visual impact.
3. *In class, or in the office*, students and I have discussed routine exercises 1–10, 12 and substantial problems 1, 2. These are either assigned or will be presented in class.
4. Cardinals of infinite well-ordered sets are called *alephs*. This terminology is due to Cantor, who introduced the sequence of cardinals $\aleph_0, \aleph_1, \aleph_2, \dots$, beginning with the smallest one. Further, he introduced $\aleph_\nu$ for each ordinal ν . According to the well-ordering theorem, which is equivalent to the axiom of choice, every set can be well-ordered, so every cardinal is an aleph. For that reason, the aleph notation is often used for cardinals in general. These notes avoid it because they avoid ordinals. In particular, ω is used here for the smallest infinite cardinal, in place of Cantor’s $\aleph_0$.