

1. *In class.* Here is the “standard” proof that $\mathbb{N} \leq U$ for every infinite set U . It’s different from the one in the *Maximal Principles* unit.
 - a. Given an infinite set U .
 - b. To find: an injection $f: \mathbb{N} \rightarrow U$.
 - c. By the axiom of choice there is a function $c: (\mathcal{P}U) - \{\emptyset\} \rightarrow U$ such that $c(S) \in S$ for every nonempty subset $S \subseteq U$.
 - d. Let u be any element of U .
 - e. Define $d = c \cup \{\langle \emptyset, u \rangle\}$.
 - f. Then $d: \mathcal{P}U \rightarrow U$ and $d(S) \in S$ for every nonempty $S \subseteq U$.
 - g. Define $g: \mathbb{N} \times \mathcal{P}(\mathbb{N} \times U) \rightarrow U$ by setting

$$g(\langle n, e \rangle) = d(U - \{v \in U : (\exists m \in \mathbb{N}) \langle m, v \rangle \in e \ \& \ m < n\})$$

for all $n \in \mathbb{N}$ and all $e \subseteq \mathbb{N} \times U$.

- h. By substantial problem 5 in the *Maximal Principles* unit—that is, the recursive definition theorem—since $\langle \mathbb{N}, \leq \rangle$ is well-ordered, there’s a unique function $f: \mathbb{N} \rightarrow U$ such that for each $n \in \mathbb{N}$,

$$\begin{aligned} f(n) &= g(\langle n, \{ \langle m, f(m) \rangle : m \in \mathbb{N} \ \& \ m < n \} \rangle) \\ &= d(U - \{v \in U : (\exists m \in \mathbb{N}) [v = f(m) \ \& \ m < n]\}) \\ &= d(U - \{f(m) : m \in \mathbb{N} \ \& \ m < n\}) \end{aligned}$$

- i. I claim f is injective. Suppose not.
 - i. Then there would exist a smallest $n \in \mathbb{N}$ such that $f(l) = f(n)$ for some $l \in \mathbb{N}$ with $l < n$.
 - ii. If $U = \{f(m) : m \in \mathbb{N} \ \& \ m < n\}$,
 - (1) then $\{\langle m, f(m) \rangle : m \in \mathbb{N} \ \& \ m < n\}$ would be a bijection from $\{m \in \mathbb{N} : m < n\}$ to U ,
 - (2) which is impossible because U is infinite.
 - iii. Thus $U - \{f(m) : m \in \mathbb{N} \ \& \ m < n\} \neq \emptyset$,
 - iv. so $f(n) \in U - \{f(m) : m \in \mathbb{N} \ \& \ m < n\}$,
 - v. which is impossible because $f(n) = f(l)$.