

1. *In class, Maximal Principles unit*
 - a. Routine exercise 1: the axiom of choice implies the Teichmüller–Tukey lemma.
 - i. Let $\mathcal{F}$, a family of subsets of a set X , have finite character.
 - ii. To prove: $\mathcal{F}$ has a maximal member.
 - iii. Strategy: Verify that the partially ordered set $\langle \mathcal{F}, \subseteq \rangle$ satisfies the hypotheses of the Kuratowski–Zorn principle, and use that to get a maximal member.
 - iv. So, consider a chain $\mathcal{C} \subseteq \mathcal{F}$.
 - v. To prove: $\mathcal{C}$ has an upper bound.
 - vi. This will follow if it's shown that $\bigcup \mathcal{C} \in \mathcal{F}$.
 - vii. And that follows if it's shown that $E \in \mathcal{F}$ whenever E is a finite subset of $\bigcup \mathcal{C}$.
 - viii. But such an E is a subset of some member C of $\mathcal{C}$ because $\mathcal{C}$ is a chain; thus $E \in \mathcal{F}$ because $C \in \mathcal{F}$.
 - b. Substantial problem 1: the Teichmüller–Tukey lemma implies the axiom of choice.
 - i. Given a set I and a nonempty set A_i for each $i \in I$.
 - ii. To find $m : I \rightarrow \bigcup_{i \in I} A_i$ such that $m_i \in A_i$ for each $i \in I$.
 - iii. Let $\mathcal{F}$ be the family of all injections $m \subseteq I \times \bigcup_{i \in I} A_i$ such that $m_i \in A_i$ for each $i \in \text{Dom } m$.
 - iv. Show that $\mathcal{F}$ has finite character. I'll leave out some details here.
 - v. The Teichmüller–Tukey lemma implies that $\mathcal{F}$ has a maximal member m .
 - vi. If there existed $i \in I - \text{Dom } m$, then there would exist $a \in A_i$. Set $m' = m \cup \{ \langle i, a \rangle \}$. It's straightforward to show that $m \subseteq m' \in \mathcal{F}$ and $m \neq m'$. That would contradict the maximality of m , so such an i could not exist.
 - vii. Thus $\text{Dom } m = I$, as required.
2. *About assignment 16*
 - i. Routine exercises 3–5 are applications of the maximal principle.
 - ii. Substantial problem 3 is an application of routine exercise 3.
 - iii. For finite X , Szpilrajn's theorem (routine exercise 3) is called *topological sorting*. Algorithms to carry it out are useful in data processing, and making them efficient is a major concern in computer science.
 - iv. Substantial problems 6–8 introduce one of my favorite subjects. I picked a couple of typical results whose proofs lie within your reach, and divided the work into problems that can be done independently. (Each of these problems depends some parts of the *statements* of preceding ones, but not on the steps of their solutions.)

- v. I call a nonempty family of subsets of a set X a *Moore family* if it has a maximum member and contains the intersection of each of its nonempty subfamilies. (This is not a standard term.) In the *Complete Lattices* unit it is shown that Moore families are complete lattices.
 - vi. *Filters* are often called *dual ideals*; *ultrafilters* are often called *maximal filters*.
- b. I'll discuss the proof that $\omega \preceq X$ for every infinite set X ,
- c. and the equivalence of the notions of infinite and Dedekind infinite sets.
- 3. *History*
 - a. The “standard” proof that for every infinite set U there exists an injection $f: \mathbb{N} \rightarrow U$ defines f recursively: for each $n \in \mathbb{N}$, $f(n)$ = an element, provided by the axiom of choice, of the set $U - f[\{m \in \mathbb{N} : m < n\}]$. The argument must be rearranged to permit definition of f *before* a recursive proof that the choice is from a nonempty set, and finally, to include a recursive proof that f is injective. This is the argument to which Bettazzi objected in 1896.
 - b. The proof of that result in the *Maximal Principles* unit replaces the recursions by use of the maximal principle.
 - c. Around 1940 the Polish mathematician Edward Szpilrajn changed his name to the more Polish sounding Marczewski. (The German spelling of his original name is Spielrein.) He survived World War II and had a distinguished career afterward.
 - d. Oswald Teichmüller was a leader of the Nazi students who hounded Jewish professors in Göttingen during the 1930s. He achieved some notable results, then became a soldier and went missing in action in 1943 in Russia.
 - e. John Tukey, a professor at Princeton, was a leader in American applied mathematics.