1. *Assignment 22*
   a. Read the *Cardinals II* unit. Locate and read related material in Stoll [1963] 1979.
   b. We should apportion among the class responsibility for finding and discussing solutions for routine exercises $2, 4, 7–10, 12$, and substantial problem 2.
   c. This is the hand-in assignment:
      i. two of routine exercises $1, 5, 6$;
      ii. one of substantial problems $1, 13$;
      iii. one of substantial problems 3–6;
      iv. one of substantial problems $7, 8$.
2. *Homework scores* so far.
   a. 3 students have > 80% credit on 12 problems
   b. 2 students have this on 9 problems
   c. 2 students have it on 5 or 6 problems
   d. 2 have it on < 5 problems

   For this report I didn't distinguish 1-point from 2-point problems.
3. I've posted links to the unit that
   a. comes next after the *Cardinals II* unit: *Boolean Logic*.
   b. Students have been asking in the office some questions related to that.
   c. I posted another link to the reading *Binary Addition Circuit*, a standard application of Boolean logic, which I do not regard as part of this course,
   d. and another link to *Binary Arithmetic*, some background for the application to binary addition.
   e. These three units were designed for use in Math 330, Math 301, CSc 310, and CSc 210.
4. *Maximal Principles unit, substantial problem 5*
   a. *Proposition*. Given a well-ordered set $<X, \leq>$, a set Y, and a function

      $$g : X \times \mathscr{P}(X \times Y) \to Y,$$

      there's a unique function $f : X \to Y$ such that for each $x \in X$,

      $$f(x) = g(<x, \{ <w, f(w)> : w \in X \ \& \ w < x\}>).$$

   b. Use of the maximal principle replaced classical arguments in analysis and algebra that constructed a well-ordering of some set X, usually of numbers or functions, cited the theorem that every well-ordered set is isomorphic to a set of ordinal numbers, then used recursive definitions and proofs on the ordinals to demonstrate some property P of the elements of X. Zermelo's theorem that every set can be well-ordered, based directly on the axiom of choice, justified the first step of the classical argument. This proposition, based on Bourbaki's fixpoint theorem, shows that however the well-ordering is constructed, the recursive definitions and proofs can be carried out without

referring to ordinal numbers. This permits graceful use of the recursions in cases where they reveal some particularly interesting properties besides P.

c. The statement of the proposition, for the standard well-ordered set $\mathbb{N}$, is identical to one of the common forms of recursive definition used in elementary courses.

d. *Proof.*

i. Let $\mathcal{S}$ denote the family of all *initial segments* of X: subsets $S \subseteq X$ such that $(\forall x \in S)(\forall w \in X)[w < x \Rightarrow w \in S]$.

ii. $\mathcal{S} \neq \phi$ because $\phi \in \mathcal{S}$.

iii. Let F denote the family of all functions $f \subseteq X \times Y$ such that $\operatorname{Dom} f$ is an initial segment and for all $x \in \operatorname{Dom} f$,

$$f(x) = g(<x, \{<w, f(w)> : w \in X \ \& \ w < x\}>). \qquad (*)$$

iv. If $C \subseteq F$ is a chain, then $\bigcup C \in F$. *Proof.*

(1) By basic set theory, $\bigcup C$ is a function and a subset of $X \times Y$.

(2) $\operatorname{Dom} \bigcup C$ is an initial segment. *Proof.*

(a) Let $x \in \operatorname{Dom} \bigcup C$, $w \in X$, and $w < x$.

(b) $\operatorname{Dom} \bigcup C = \bigcup_{f \in C} \operatorname{Dom} f$.

(c) $x \in \operatorname{Dom} f$ for some $f \in C$.

(d) $\operatorname{Dom} f$ is an initial segment.

(e) $w \in \operatorname{Dom} f$.

(f) $w \in \bigcup_{f \in C} \operatorname{Dom} f = \operatorname{Dom} \bigcup C$.

(3) Let $x \in \operatorname{Dom} \bigcup C = \bigcup_{f \in C} \operatorname{Dom} f$. Then (*) holds. Moreover,

(a) If $w \in X$ and $w < x$, then $<w, f(w)> \in f \subseteq \bigcup C$, so that $f(w) = [\bigcup C](w)$. By (*),

$$f(x) = g(<x, \{<w, [\bigcup C](w)> : w \in X \ \& \ w < x\}>).$$

(b) That is, f contains the ordered pair whose left and right entries are x and the right-hand side of the previous equation. Thus, that pair belongs to $\bigcup C$:

$$[\bigcup C](x) = g(<x, \{<w, [\bigcup C](w)> : w \in X \ \& \ w < x\}>).$$

v. Define a function $G : F \to F$ as follows.

(1) For each $f \in F$ set $G(f) = f$ if $\operatorname{Dom} f = X$.

(2) Otherwise, let x be the minimum element of $X - \operatorname{Dom} f$, note that $w \in X \ \& \ w < x \Rightarrow w \in \operatorname{Dom} f$ and thus $(\operatorname{Dom} f) \cup \{x\}$ is an initial segment, set

$$y = g(<x, \{<w, f(w)> : w \in X \ \& \ w < x\}),$$
$$f' = f \cup \{<x, y>\},$$

and note that $w \in X \ \& \ w < x \Rightarrow f'(w) = f(w)$, hence

$$f'(x) = g(<x, \{<w, f'(w)> : w \in X \ \& \ w < x\}).$$

Therefore, $f' \in F$. Set $G(f) = f'$.

vi. Clearly, $f \subseteq G(f)$ for every $f \in F$.

vii. By the Bourbaki fixpoint theorem, there exists $f \in F$ for which $G(f) = f$.

viii. $\mathrm{Dom} f = X$ by definition of G.