

1. *In class, the Maximal Principles unit*
 - a. Some authors call the empty set a chain, too. That requires slightly different statements of some theorems.
 - b. I showed that sets M, M' in the proof of the Bourbaki fixpoint theorem are admissible.
 - c. The proof that M , the intersection of all admissible sets, is admissible, is an example of impredicative reasoning: the family of all admissible sets must be manipulated before one of its elements M is defined. There has been much philosophical discussion about whether impredicativity should be allowed. Virtually all mathematicians ignore, are in fact unaware, of the controversy.
 - d. I solved routine exercise 6, but in writing these notes discovered a flaw in the problem. It should be stated this way:
 - e. if $\langle X, \leq \rangle$ is a partially ordered set *with minimum element* o , in which every chain has a supremum, then every homomorphism f of X has a fixpoint.
 - f. This is not true without the italicized assumption previously omitted: consider a two-element set partially ordered by the identity relation, and the function that interchanges its elements.
 - g. I ran afoul of the definition of chain: I'd copied the statement from a source that regarded the empty set as a chain. A supremum of the empty set is automatically the minimum element of X .
 - h. Given the correct hypothesis, consider the set $X' = \{x \in X : x \leq f(x)\}$.
 - i. Then $\langle X', \leq \rangle$ is a partially ordered set since $o \in X'$.
 - ii. Every chain C in X' has a supremum in X .
 - iii. Since $x \in C \Rightarrow [x \in X' \ \& \ x \leq \bigvee C] \Rightarrow x \leq f(x) \leq f(\bigvee C)$, it follows that $\bigvee C \leq f(\bigvee C)$, and thus $\bigvee C \in X'$.
 - iv. Further, $x \in X' \Rightarrow x \leq f(x) \Rightarrow f(x) \leq f(f(x)) \Rightarrow f(x) \in X'$.
 - v. Thus $\langle X', \leq \rangle$ and f satisfy the hypotheses of the Bourbaki fixpoint theorem: f has a fixpoint in X' , and thus one in X .
2. *History*
 - a. See Moore 1982, §4.4, for a history of the maximal principle. Hausdorff (1914) and Kuratowski (1922) regarded it as a theorem, to be deduced from the well-ordering theorem. But it remained cloudy, and Zorn (1935) regarded it as an axiom. His paper and some earlier lectures showed its utility, and the principle is thus commonly called *Zorn's Lemma*.
 - b. Before Kuratowski (1922), the well-ordering theorem was heavily used in algebra and analysis, usually to permit very messy recursive definitions and proofs using ordinal number theory, which itself is rather messy. Kuratowski localized the mess in his proof of the maximal principle: it could be used neatly to replace the recursive definitions and use of ordinals in algebra and analysis. But those methods remained in the proof of the maximal principle until Bourbaki replaced them with his fixpoint theorem.

- c. Click here for portraits of [Hausdorff](#), [Kuratowski](#), and [Bourbaki](#).
 - d. This theorem is closely involved with European politics. Zorn participated in the great development of algebra at Göttingen during the 1920s. Bourbaki was the name of a fervently patriotic French group who were countering that German dominance. Kuratowski was one of the first students of the highly politically motivated Polish mathematical renaissance which was emerging after World War I from Russian oppression. He helped run an “underground” mathematical school in Poland during its World War II occupation by the Nazis. Zorn, luckily, emigrated to the United States in time to escape the Nazis. Hausdorff did not: he and his wife committed suicide as they were about to be transported to the extermination camps.
3. *Projects*
- a. *For a term paper.* Investigate further and report on the controversy stimulated by Zermelo [1904] 1970. Most of the original materials are now available in English.
 - b. *Probably for a master’s expository paper.* The ultrafilter theorem leads to a representation theorem: every distributive lattice is isomorphic to a lattice of sets. There is a considerable literature connecting these theorems, similar ones for other algebraic structures such as Boolean algebras, representations as lattices of particular kinds of sets or functions, various forms of the axiom of choice, and various major theorems in general topology and functional analysis. Most of that literature is in English, and is at the same level as the solution of substantial problem 8. This work started with that of Marshall Stone and Garrett Birkhoff in the 1930s. I suspect that several projects could be fashioned from a survey of some of that literature.
 - c. *For a term paper or a master’s expository paper.* Kanamori 1997, 305, refers to versions of Bourbaki’s fixpoint theorem used by Dana Scott and others to develop denotational semantics for programming languages. This is intriguing: Scott was one of my teachers, both in logic and computer science, but before he did that work. I’ve never followed up on it, and would welcome someone’s explaining it to me.
 - d. I’d not encountered Kanamori’s work before preparing for this course. I notice that a number of his recent expository papers on set theory are reviewed in MSN. Perhaps one of them could provide leads for a project.