

1. *In class, the Axiom of Choice unit*
 - a. We discussed the trivial questions and routine exercise 1.
 - b. Routine exercise 2 is concerned with the proof that a real-valued function f of a real variable x is continuous at an argument x if and only if it is sequentially continuous there. This theorem is used routinely in calculus classes, proved in real-analysis courses, and was one of the first places where a tacit use of the axiom of choice was noticed and criticized. Here's the proof.
 - i. Suppose f is continuous at x .
 - (1) To prove: f is sequentially continuous at x .
 - (2) Consider a sequence $c_0, c_1, \dots$ that converges to x .
 - (3) To prove: the sequence $f(c_0), f(c_1), \dots$ converges to $f(x)$.
 - (4) Given $\varepsilon > 0$.
 - (5) To find: m such that $|f(c_n) - f(x)| < \varepsilon$ whenever $n > m$.
 - (6) Since f is continuous at x there exists $\delta > 0$ such that $|f(c) - f(x)| < \varepsilon$ whenever $|c - x| < \delta$.
 - (7) By (2) there exists m such that $|c_n - x| < \delta$ whenever $n > m$.
 - (8) This m satisfies the requirement in (5).
 - ii. Suppose f is sequentially continuous at x .
 - (1) To prove: f is continuous at x .
 - (2) Given $\varepsilon > 0$.
 - (3) To find: $\delta > 0$ such that $|f(c) - f(x)| < \varepsilon$ whenever $|c - x| < \delta$.
 - (4) All we know is something about sequences that converge to x . That doesn't seem close to the problem of finding a δ ; we don't have enough to start with. In such a case it often helps to switch the approach and prove the converse of the result at hand. So let's start over.
 - iii. Suppose f is not continuous at x .
 - (1) To prove: f is not sequentially continuous at x .
 - (2) That is, we must find a sequence $c_0, c_1, \dots$ that converges to x , such that the related sequence $f(c_0), f(c_1), \dots$ does not converge to $f(x)$.
 - (3) Since f is not continuous at x , there exists $\varepsilon > 0$ such that for every $\delta > 0$ there exists c such that $|c - x| < \delta$ but $|f(c) - f(x)| \geq \varepsilon$.
 - (4) For each $n \in \mathbb{N}$ the set $A_n = \{c \in \mathbb{R} : |c - x| < 1/(n+1)\}$ is non-empty. By the axiom of choice, there is a choice function $c \in X_n A_n$. The sequence c satisfies the requirement in (2): use subscript notation for its values, so that $c_n \in A_n$ for all n .
 - c. Such uses of the axiom of choice are often tacet: mathematicians just write in place of (4) something like "let c_n be such a c for $\delta = 1/(n+1)$ " without

mentioning the axiom. On 7 March 2008, for example, Prof. Ardila did so in his Math 490 lecture on Coxeter groups. He had no reason to call attention to the axiom.

- d. But students of foundations should be aware that making infinitely many choices without specifying a rule for doing so actually *requires* some set-theoretic principle beyond those covered in the *Basic Set Theory Unit*.