

1. *Assignment 16: Maximal Principles unit*
 - a. any two of routine exercises 3–5;
 - b. any two of substantial problems 2–4, 6–8.
2. *In class, Cardinals I*
 - a. We discussed proposition 32, that $\alpha^{\beta+\gamma} = \alpha^\beta \alpha^\gamma$ for any cardinals α, β, γ . It's enough to show that $A^{B' \cup C'} \approx A^B \times A^C$ for any sets A, B, C , where $B' = B \times \{0\}$ and $C' = C \times \{1\}$. Since $B \approx B'$ and $C \approx C'$, it suffices to show $A^{B' \cup C'} \approx A^{B'} \times A^{C'}$. To this end, consider the functions $d: A^{B' \cup C'} \rightarrow A^{B'} \times A^{C'}$ and $f: A^{B'} \times A^{C'} \rightarrow A^{B' \cup C'}$ defined by setting $d(e) = \langle e \cap (B' \times A), e \cap (C' \times A) \rangle$ and $f(\langle g, h \rangle) = g \cup h$ for all $e \in A^{B' \cup C'}$, $g \in A^{B'}$, and $h \in A^{C'}$. They're each other's inverses, so are bijective.
 - b. The substance of routine exercise 2 is to prove that for any $n \in \mathbb{N}$, a function f from $\{k \in \mathbb{N} : k < n\}$ to itself is injective if and only if it's surjective.
 - i. First, assume f is injective. Were f not surjective, $(\exists m < n)(\forall l < n) f(l) \neq m$, so that $n > 0$ and the function e from $\{k \in \mathbb{N} : k < n\}$ to itself defined by setting $e(l) = f(l)$ if $l < m$, $= n - 1$ if $l = m$, and $= f(l) - 1$ if $l > m$ would be bijective, and $e \circ f$ would be an injection from $\{k \in \mathbb{N} : k < n\}$ to $\{k \in \mathbb{N} : k < n - 1\}$, and hence $n \leq n - 1$ by an earlier result, contradiction! Thus injectivity implies surjectivity.
 - ii. Now suppose f is surjective. Then for each $m \in \mathbb{N}$ with $m < n$ there exists $l \in \mathbb{N}$ such that $f(l) = m$; let $e(m) =$ the first such l . This defines a function e from $\{k \in \mathbb{N} : k < n\}$ to itself, such that $f(e(m)) = m$ for each such m . If also $m' \in \mathbb{N}$, $m' < n$, and $e(m) = e(m')$, then $m = f(e(m)) = f(e(m')) = m'$; that is, e is injective. By part (i), e is surjective. Now suppose $l, l' \in \mathbb{N}$ and $f(l) = f(l')$; since e is surjective, $l = e(m)$ and $l' = e(m')$ for some $m, m' \in \mathbb{N}$ with $m, m' < n$, hence $m = f(e(m)) = f(l) = f(l') = f(e(m')) = m'$. That is, f is injective.
 - c. Later it will be shown that only finite sets U have the property that all functions from U to U are injective if and only if they're surjective. That will be used to show that the finiteness of the universe of discourse can be expressed in second-order logic, even though that is not so in first-order logic. In both types of logic, one can express the injectiveness or surjectiveness of a single function. First-order logic does not permit quantification over functions, but second-order logic does.