

1. *Grading*

- a. I'm planning to assign about 85 points to homework problems altogether, constituting 55% of the course, with this partial scheme:
 - i. *Basic Set Theory* trivial questions 2 points altogether
 - ii. Routine exercises 1 point each
 - iii. Substantial problems 2 points each
 - iv. Assignment 2d 2 points
 - v. Stoll 1.5.6 2 points
 - vi. Tarski exercises 1 point each
- b. I've planned assignments on the units posted on the course website; they continue in much the same fashion as those on the units we've already covered. But I've not yet planned assignments for the subsequent material on Boolean and elementary logic. I expect those to continue in much the same way, except that they'll be more routine, not involving such delicate concepts.
- c. I generally assign letter grades according to the following scheme:

80% ...	$A-, A$
65% ... 79%	$B-, B, B+$
50% ... 64%	$C-, C, C+$
40% ... 49%	$D-, D, D+$
... 39%	F

(Sometimes unforeseen circumstances may force a change in this.)
- d. Later, when we discuss details of the term paper, I'll describe how I've graded those.

2. *More on definitions.*

- a. You may still be puzzled: why can't you just define $\#A \leq \#B$ to mean $A \preceq B$, without bothering first to prove proposition 8 in the *Cardinals I* unit? The problem is perhaps clarified by regarding it as that of defining " $\alpha \leq \beta$ ". The *Cardinals I* definition would be $(\exists A, B)[\alpha = \#A \ \& \ \beta = \#B \ \& \ A \preceq B]$.
- b. With that definition, I can prove $(\forall \text{ cardinals } \alpha)[\alpha \leq \alpha]$ without using proposition 8, as follows.
 - i. Given a cardinal α
 - ii. To prove $\alpha \leq \alpha$
 - iii. By step 1, $\alpha = \#A$ for some A .
 - iv. By a previous proposition $A \preceq A$.
 - v. By steps 3, 4 and the *Cardinals I* definition, $\alpha \leq \alpha$.
- c. Now try to prove $(\forall \text{ cardinals } \alpha, \beta, \gamma)[\alpha \leq \beta \ \& \ \beta \leq \gamma \Rightarrow \alpha \leq \gamma]$. Notice that I don't really need the quantifier $(\forall \text{ cardinals } \alpha, \beta, \gamma)$ because its information is given by the hypothesis $\alpha \leq \beta \ \& \ \beta \leq \gamma$. Nevertheless, I'll put it in because it seems to clarify the statement to be proved. You'll see that it plays no role in the proof. (This was not the case in item b.)
 - i. Given cardinals α, β, γ

- ii. Given $\alpha \leq \beta$
- iii. Given $\beta \leq \gamma$
- iv. To prove $\alpha \leq \gamma$
- v. By step 1, $\alpha = \#A'$, $\beta = \#B'$, and $\gamma = \#C'$ for some sets A', B', C' . (I won't need to refer to these again.)
- vi. By step 2, $(\exists A, B)[\alpha = \#A \ \& \ \beta = \#B \ \& \ A \preceq B]$. You have to use new variables because A', B', C' have already been introduced.
- vii. By step 3, $(\exists B'', C)[\beta = \#B'' \ \& \ \gamma = \#C \ \& \ B'' \preceq C]$.
- viii. By steps 6, 7, $\#B = \#B''$, hence $B \approx B''$.
- ix. $A \approx A \ \& \ B \approx B'' \ \& \ A \preceq B$ by a previous proposition and steps 6, 8.
- x. By proposition 10, $A \preceq B''$.
- xi. By steps 7, 10, $A \preceq C$.
- xii. $\alpha = \#A \ \& \ \gamma = \#C \ \& \ A \preceq C$ by steps 6, 7, 11.
- xiii. $\alpha \leq \gamma$ by step 12 and the *Cardinals I* definition.
- d. Steps 1 and 5 and the comment in step 6 can be omitted from the previous proof to get a proof of the unquantified formula $\alpha \leq \beta \ \& \ \beta \leq \gamma \Rightarrow \alpha \leq \gamma$. But you've still cited a version of proposition 10.
- e. You might think that $(\forall A, B)[\alpha = \#A \ \& \ \beta = \#B \Rightarrow A \preceq B]$ would be a suitable alternative definition for " $\alpha \leq \beta$ ". But no, that would make the latter statement true whenever α and β are not cardinals, which is probably not what you intend.
- f. Compromise, try: $(\exists A, B)[\alpha = \#A \ \& \ \beta = \#B \ \& \ (\forall A', B')[\alpha = \#A' \ \& \ \beta = \#B' \Rightarrow A' \preceq B']]$.

I think this will let you avoid using proposition 10 until you need to prove $A \preceq B \Rightarrow \#A \leq \#B$, but you do need it there because the conclusion affirms $A' \preceq B'$ for all A', B' equinumerous with A, B .

3. *Finiteness*

- a. The simple and natural definition of finiteness given here presupposes that the natural number system $\mathbb{N}$ has been built into the set theory under development.
- b. *Infinite* simply means *not finite*.
- c. Another simple definition, perhaps not quite so natural, given by Dedekind in [1888] 1963, does not presuppose the natural number system. But with it, proving some elementary properties of finiteness requires the axiom of choice. We'll get to that very soon.
- d. A third definition, given by Tarski in 1924a, involves neither the natural number system nor the axiom of choice, but seems very unnatural. We won't deal with that one.
- e. The proof that $\mathbb{N}$ is infinite, in the *Cardinals I* unit just after proposition 15, is needlessly complex. Here's a shorter one: if $\mathbb{N}$ were finite, there would exist a bijection $f: \mathbb{N} \rightarrow \{m \in \mathbb{N} : m < n\}$ for some $n \in \mathbb{N}$, and $f \circ g: \{m \in \mathbb{N} : m < n+1\} \rightarrow \{m \in \mathbb{N} : m < n\}$ injectively, contrary to proposition 8, where $g: \{m \in \mathbb{N} : m < n+1\} \rightarrow \mathbb{N}$ is the identity function.

- f. To prove proposition 16 recursively, cast it in the form $(\forall n \in \mathbb{N})P_n$, where P_n is the proposition $(\forall m \in \mathbb{N})(\forall A, B)[A \cap B = \phi \ \& \ \#A = m \ \& \ \#B = n \Rightarrow \#(A \cup B) = \#A + \#B]$.
- i. P_0 says essentially $(\forall m \in \mathbb{N})\forall A[\#A = m \Rightarrow \#(A \cup \phi) = \#A + 0]$, which is true.
 - ii. Let $n \in \mathbb{N}$.
 - iii. Assume P_n .
 - iv. It will suffice to prove P_{n+1} . To that end,
 - v. Let $m \in \mathbb{N}$,
 - vi. suppose A, B are sets, and
 - vii. $A \cap B = \phi \ \& \ \#A = m \ \& \ \#B = n + 1$.
 - viii. To prove: $\#(A \cup B) = \#A + \#B$.
 - ix. There are bijections $f: \{k \in \mathbb{N} : k < m\} \rightarrow A$ and $g: \{k \in \mathbb{N} : k < n + 1\} \rightarrow B$.
 - x. Let $b = g(n)$, so that $b \in B \ \& \ b \notin A$.
 - xi. Let $A' = A \cup \{b\}$ and $f' = f \cup \{<m, b>\}$.
 - xii. Then $f': \{k \in \mathbb{N} : k < m + 1\} \rightarrow A'$ bijectively.
 - xiii. Let $B' = B - \{b\}$ and $g' = g - \{<n, b>\}$.
 - xiv. Then $g': \{k \in \mathbb{N} : k < n\} \rightarrow B'$ bijectively.
 - xv. So $\#A' = m + 1$, $\#B' = n$, and $A \cup B = A' \cup B'$.
 - xvi. By P_n , $\#(A' \cup B') = (m + 1) + n$.
 - xvii. Thus $\#(A \cup B) = (m + 1) + n = m + (n + 1) = \#A + \#B$.
- g. The *inclusion-exclusion formula*, familiar to students of combinatorics, shows how to compute the cardinal of the union of several finite sets. Segercrantz 1998 presents an elegant and well-motivated statement and proof of that theorem.