

1. *Assignment 12*
 - a. The unproved propositions in the *Cardinals I* unit. We have apportioned among the class responsibility for finding and discussing proofs for propositions 9–15, 18–19, 24–26, 32. I’m still looking for some to do propositions 34–36.
 - b. You should also be prepared to discuss a solution of its routine exercise 2.
 - c. Read the *Axiom of Choice* unit.
 - i. Locate and read related material in Stoll [1963] 1979.
 - ii. Be prepared to discuss its trivial questions, routine exercises 1, 2, and substantial problem 5.
 - d. Read Smoryński’s and Rubin’s reviews of Moore 1982. They’re online.
2. *In class, Cardinals I unit*
 - a. I discussed the proof of the Cantor–Bernstein theorem and the (undefined) notion of cardinal.
 - b. A discussion of the theorem $\#X = \#X' \ \& \ \#Y = \#Y' \ \& \ X \leq Y \Rightarrow X' \leq Y'$ was confusing.
 - i. The problem was attempted inferences of $\#X \leq \#Y$ from $X \leq Y$ and/or the converse inference with X', Y' . Neither step is valid because
 - (1) the relation $\leq$ between cardinals hadn’t been defined at the point where the theorem was stated, and
 - (2) that relation can’t be defined as $\#X \leq \#Y \Leftrightarrow X \leq Y$ until that theorem is proved.
 - ii. Since the two cardinals are cardinals of sets X, Y for many choices of those sets, you must know that the truth of $X \leq Y$ is independent of which sets are referred to.
 - iii. Thus, this theorem is what permits the definition: you can’t use the definition to prove the theorem.
 - iv. The proof is straightforward: the three premisses imply existence of bijections $f : X \rightarrow X'$, $g : Y \rightarrow Y'$ and an injection $h : X \rightarrow Y$, so that $g \circ h \circ f^{-1} : X' \rightarrow Y'$ injectively.
 - c. Definitions are hard. In 1904, for instance, Oswald Veblen proposed a definition of equidistance in terms of betweenness in Euclidean geometry. In 1907, Federigo Enriques criticized the validity of that definition. Not until 1935 was the matter settled, by Tarski, who proved that using elementary logic, equidistance *cannot* be defined in terms of betweenness. Tarski’s proof is quite simple, but the concepts are delicate and needed to be described very precisely in his general theory of definitions.
3. *History*
 - a. Georg Cantor introduced the $\leq$ and $\approx$ relations in [1878, page 242](#), and used them to study many examples now familiar.

- b. In [1884, page 388](#), Cantor stated the Cantor–Bernstein theorem, but offered no proof. By 1887, Richard Dedekind had found a proof, but did not announce it. (See [Dedekind 1930–1932a](#).) In 1896, Cantor’s assistant Felix Bernstein provided one, which Cantor publicized informally at the 1897 first International Congress of Mathematicians in Zürich. Émile Borel published that proof in [1898, 102ff](#), acknowledging Bernstein. That same year, the logician Ernst Schröder independently published an argument for the theorem, but his was eventually found inadequate. Nevertheless, the result has often been called the *Schröder*–Bernstein theorem. The proof given here follows the route of Tarski 1955, but probably stems from his work in the 1920s. These details are discussed in Grattan-Guinness 2000, §4.2.5, 132–135, and in Frewer 1981, 84–86.
 - c. Click here for a picture of [Cantor](#) or [Bernstein](#).
4. *Project for a term paper.* Report more details about the Cantor–Bernstein theorem, for example: various proofs, faulty arguments, related theorems, and applications. You can get started by consulting references cited in these notes and using JSTOR to search the full text of various journals for (Cantor OR Schroeder) AND Bernstein. A related project was discussed in outline 9.