

1. *Assignment 11*
 - a. Tarski [1946] 1995, chapter 5, the part of exercise 3 on page 110.
 - b. Tarski [1946] 1995, chapter 5, exercise 8.
 - c. The unproved propositions in the *Cardinals I* unit. We have apportioned among the class responsibility for finding and discussing proofs for propositions 9–15, 18–19, 24–26, 30, 32, 34–36.
 - d. You should also be prepared to discuss a solution of its routine exercise 2.
2. *In class* I'll respond to questions about Tarski [1946] 1995, chapter 5: in particular, its exercises 2, the part of 3 on page 109, and exercises 6, 9.
3. *Cardinal arithmetic*. These notes develop basic properties of the $\leq$ relation between cardinals and the operations of cardinal addition, multiplication, and exponentiation. Three types of theorems are required for each of these.
 - a. First, a theorem describes the situation for finite sets. This is presumably familiar, but a proof from more elementary principles is not. Such proofs require elementary recursion techniques. I'll sketch three of them, and ask you to provide details for the fourth.
 - b. Second, a theorem prepares the way for definition of the cardinal relation or operation. Since the definition refers to sets with given cardinals, the theorem must prove that the truth of the relation or outcome of the operation does not depend on which sets are referred to. I'll sketch proofs of two of them, and ask you to provide details for the others.
 - c. Third, various theorems state the algebraic rules of cardinal arithmetic. I'll sketch proofs of some of them, and ask you to provide details for some others.
4. *In class, Cardinals I unit*
 - a. I'll respond to questions about the *Cardinals I* unit.
 - b. Here's its proposition 8:

$$(\forall n \in \mathbb{N})(\forall m \in \mathbb{N})[\{k \in \mathbb{N} : k < m\} \preceq \{k \in \mathbb{N} : k < n\} \Rightarrow m \leq n].$$

Proof. The desired result is $(\forall n \in \mathbb{N})P_n$, where

$$P_n \text{ is } (\forall m \in \mathbb{N})[\{k \in \mathbb{N} : k < m\} \preceq \{l \in \mathbb{N} : l < n\} \Rightarrow m \leq n].$$

I'll suppose the desired result is false, and obtain a contradiction. Thus I'm supposing P_n is false for some n . Select the *first* such n , so that

$$\neg (\forall m \in \mathbb{N})[\{k \in \mathbb{N} : k < m\} \preceq \{l \in \mathbb{N} : l < n\} \Rightarrow m \leq n].$$

This is equivalent to

$$(\exists m \in \mathbb{N})[\{k \in \mathbb{N} : k < m\} \preceq \{l \in \mathbb{N} : l < n\} \ \& \ n < m].$$

That is,

$$(\exists m \in \mathbb{N})(\exists f: \{k \in \mathbb{N} : k < m\} \rightarrow \{l \in \mathbb{N} : l < n\}) \\ [n < m \text{ \& } f \text{ is injective}].$$

Then $m > 0$ because $m > n$, so $\text{Dom } f \neq \emptyset$, hence $\text{Rng } f \neq \emptyset$, $n > 0$, and $n - 1$ exists. Moreover, $\text{Rng } f \not\subseteq \{l \in \mathbb{N} : l < n - 1\}$, because otherwise, $f: \{k \in \mathbb{N} : k < m\} \rightarrow \{l \in \mathbb{N} : l < n - 1\}$ injectively, and then P_{n-1} would imply $m \leq n - 1$, which is impossible. Thus there exists $k < m$ such that $f(k) = n - 1$. Since f is injective there is only one such k , and the function g with domain $\mathbb{N}$ such that

$$\begin{aligned} g(j) &= f(j) && \text{if } j < k, \text{ else} \\ &= f(j + 1) && \text{if } j < m - 1, \end{aligned}$$

is an injection from $\{j \in \mathbb{N} : j < m - 1\}$ to $\{l \in \mathbb{N} : l < n - 1\}$. Now, P_{n-1} would imply $m - 1 \leq n - 1$, hence $m \leq n$, which is impossible.