

1. *Assignment 10*
 - a. Tarski [1946] 1995, chapter 2, exercise 5b. (This escaped from an earlier assignment!)
 - b. Tarski [1946] 1995, chapter 4, exercise 16. Also, determine how many vertices such a polygon could have.
 - c. Read Tarski [1946] 1995, chapter 5. In particular, be prepared to discuss its exercises 2, the part of 3 on page 109, and exercises 6, 9.
2. *In class*
 - a. I discussed the notion of a lattice and the *Complete Lattices* unit's substantial problem 1.
 - i. In short, you can either define a lattice as a partially ordered set in which each two-element subset $\{x, y\}$ has a supremum $x \vee y$ (x join y) and an infimum $x \wedge y$ (x meet y), then prove that joins and meets enjoy certain familiar properties, or you can define a lattice as an algebra with two binary operations satisfying those properties, then define the order relation in terms of join or meet, and prove that it is in fact a partial ordering with respect to which the joins and meets are the suprema and infima of the two-element sets.
 - ii. Moreover, if you start with a lattice defined the first way, define joins and meets, and then the order relation, you get back the same order relation that you started with.
 - iii. The *join* and *meet* terminology stems from Italian projective geometry books of a century ago. Hasse diagrams should convince you that the term *lattice* is apt; the French equivalent is *treillis*.
 - b. I discussed *assignment 9* briefly.
 - c. I'll discuss Moore families and closures only later, when they are required for some further considerations.
 - d. I responded to questions about Tarski [1946] 1995, chapter 4, particularly its exercises 4, 7, 9, 10, 12, 17.
3. *Cardinals*
 - a. Cardinals are usually introduced somewhat later in the development of set theory, as follows.
 - i. Basics of the theory of well-ordered sets $\langle X, \leq \rangle$ are presented.
 - ii. Ordinal numbers are defined as certain kinds of sets.
 - iii. It is shown that the relation $\alpha \leq \beta$ defined as $\alpha \in \beta \vee \alpha = \beta$ well-orders any set of ordinal numbers.
 - iv. It is shown that any well-ordered set X is equinumerous with some ordinal, and that the ordinals equinumerous with X constitute a set.
 - v. The axiom of choice is introduced, and used to show that every set X has some well-ordering, hence is equinumerous with some ordinal.
 - vi. $\#X$ is defined as the smallest ordinal equinumerous with X .

- b. By regarding $\#X$ as a basic notion, rather than a defined one, these notes avoid the theories of well-ordered sets and ordinals and the refined techniques of recursive proof and definition that those require.
 - c. $\#X$ can in fact be defined even *without* introducing the axiom of choice, through use of the notion of *rank* of a set. That route nevertheless makes major use of refined recursion techniques.
- 4. *Finiteness*
 - a. These notes regard the system $\mathbf{N}$ as a basic notion, rather than a defined one, in order to permit the definition of finiteness used here.
 - b. $\mathbf{N}$ *can* in fact be defined as a certain set of ordinals, without introducing special axioms. But deriving its standard properties requires major use of refined recursion techniques. These notes avoid those.
 - c. Sometimes a set is defined to be finite just when it is not equinumerous with any proper subset. That definition has the advantages of not requiring prior introduction of $\mathbf{N}$, and being rather easy to visualize. Proving it equivalent to the definition used here, however, *requires* use of the axiom of choice, so that notion of finiteness, often called *Dedekind finiteness*, is slightly different. It will be studied later.
 - d. Sometimes a set X is defined to be finite just when any nonempty family of subsets of X has a member that does not properly contain any other member. That definition is indeed equivalent to the one used here (the proof of equivalence doesn't require the axiom of choice). It has the advantage of not requiring prior introduction of $\mathbf{N}$, but seems nonintuitive because we do learn to count before we learn to think about finiteness.
- 5. *History*
 - a. The idea of introducing cardinals as a basic notion, rather than a defined one, stems from [Tarski 1924b, 147](#). The definition of finiteness mentioned earlier, that doesn't involve either the natural numbers or the axiom of choice, stems from [Tarski 1924a, 46](#). I learned this material from Suppes 1960, chapter 4.
 - b. Excellent books on Cantor and the history of set theory: Dauben 1979, Ferreirós Domínguez 1999, Grattan-Guinness 2000, Moore 1982.