

1. *Assignment 6*
 - a. Read Tarski [1946] 1995, chapter 2 through section 11. In particular, be prepared to discuss its exercises 3, 4, 6, 7, 9, 10.
 - b. Because we're not progressing as fast as I'd tentatively scheduled, I've moved to the next outline the rest of the assignment that was originally placed here.
2. *In class* I provided some more detail about the *Partially Ordered Sets* unit.
 - a. Contemplation of the definitions of homomorphism for partially ordered sets and for groups, rings, etc., might lead to a question whether the $\Rightarrow$ in the definition for partially ordered sets should really be $\Leftrightarrow$.
 - i. Here's a homomorphism that doesn't satisfy the definition with $\Leftrightarrow$ substituted for $\Rightarrow$: map each element of a partially ordered set with Hasse diagram  to the element on the left.
 - ii. Here's another, that's injective: map the left and right elements of that partially ordered set to the top and bottom elements of one with Hasse diagram .
 - iii. Thus, it makes a difference which version of the definition of homomorphism you use. In fact, authors have used both: take care when reading. The one used here is the more common.
 - b. If $\langle X, \leq \rangle$ is a quasi-ordered set, the relation $E = \leq \cap \check{\leq}$ is an equivalence on X . *Proof.*
 - i. Reflexivity: $I_X \subseteq \leq$, so $I_X = I_X \check{\subseteq} \check{\leq}$, so $I_X \subseteq \leq \cap \check{\leq} = E$.
 - ii. Symmetry: $E =$ (the converse of an intersection is the intersection of the converses, so) $\check{\leq} \cap \leq = \leq \cap \check{\leq} = E$.
 - iii. Transitivity: given $x E y$ and $y E z$, to prove $x E z$. From the former, $x \leq y$ & $y \leq x$. From the latter, $y \leq z$ & $z \leq y$. From the first and third, $x \leq z$. From the second and fourth, $z \leq x$. From these two, $x E z$.
 - c. In the *Partially Ordered Sets* unit a relation $\leq/E$ is defined on X/E so that $x/E \leq/E y/E \Leftrightarrow x \leq y$ for all $x, y \in X$. This method of defining a relation on a quotient family is extremely common in algebra.
 - d. $\langle X/E, \leq/E \rangle$ is a partially ordered set. *Proof.*
 - i. Reflexivity: substitute x for y in the previous $\Leftrightarrow$ statement.
 - ii. Weak antisymmetry: if $x/E \leq/E y/E$ and $y/E \leq/E x/E$ then $x \leq y$ & $y \leq x$, hence $x E y$, hence $x/E = y/E$.
 - iii. Transitivity: if $x/E \leq/E y/E$ and $y/E \leq/E z/E$ then $x \leq y$ and $y \leq z$, hence $x \leq z$ hence $x/E \leq/E z/E$.
 - e. The quotient map is a homomorphism from $\langle X, \leq \rangle$ to $\langle X/E, \leq/E \rangle$ because $x \leq y$ implies $x/E \leq/E y/E$.
 - f. You've seen the previous arguments before in algebra courses: they underly the construction of quotient structures related to groups, rings, and vector spaces. In fact, they apply to a wide variety of structures with any number

of specified operations, constants, and relations. The general setting is *model theory*.

- g. The mapping $x \mapsto x \downarrow$ is a homomorphism from any quasi-ordered set $\langle X, \leq \rangle$ to $\langle \mathcal{P}X, \subseteq \rangle$. *Proof.* Suppose $x \leq y$. To prove: $x \downarrow \subseteq y \downarrow$. That results from $t \in x \downarrow \Rightarrow t \leq x \Rightarrow t \leq y \Rightarrow t \in y \downarrow$.
- h. If $\leq$ is a partial ordering, then $x \mapsto x \downarrow$ injectively. *Proof.* If $x \downarrow = y \downarrow$, then $x \leq x \Rightarrow x \in x \downarrow \Rightarrow x \in y \downarrow \Rightarrow x \leq y$, and similarly, $y \leq x$; by weak anti-symmetry, $x = y$.
- i. Older texts often use the term *simply ordered* for *linearly ordered*.
- j. Substantial problem 2 formalizes the process of making a new partially ordered set $X_0 + X_1$ from two given ones X_0 and X_1 by setting their Hasse diagrams side by side with no connecting links. The set $X_0 + X_1$ seems to be their union. But that doesn't work when they have any common element. We're forced either to do a preliminary step in every application of this idea to assure that these sets are disjoint, or else to employ a standard programming technique to do that once and for all. I chose the latter. Namely, replace X_0 and X_1 by $X_0 \times \{0\}$ and $X_1 \times \{1\}$, respectively, then construct the union. That complicates formulation and proof of the basic properties of this construct, but simplifies their application. For instance, $X_0 + X_1$ is not equal to $X_1 + X_0$, but isomorphic to it. The isomorphism is the map $\langle x, i \rangle \mapsto \langle x, 1-i \rangle$, which reverses the tags. That's the *commutativity* property. The proof of the corresponding *associativity* property is more complex because the isomorphism must rearrange some double tags.