

1. Ms Singh reported on her term project, Hindu mathematics. This is hard for me to describe, and I won't try here. The subject is huge, and not yet well studied. Major research in ancient Indian science and mathematics is still going on. Because those developed extensively in isolation from the West, their practices often differ from ours in the extreme, but achieve comparable results. It is really difficult to describe the relationships between their techniques and our algebra, for example. Hindu mathematicians also pursued geometrical theory, analogous to or continuing Greek discoveries, which underlay important aspects of the design of temples.
2. Mr. Carlos reported on his project about Fibonacci (Leonardo of Pisa, 1170–1250) and his numbers. Fibonacci benefited from traveling with his father, an international businessman and diplomat for the republic of Pisa. He encountered much mathematical lore, especially the Hindu-Arabic numeration system, then wrote several books which were widely disseminated, introducing the West to a vastly more efficient means of computation, used both in commerce and science. In one of those books he introduced his number sequence, which starts with zero and one and continues indefinitely with each one being the sum of the previous two. Originally a curiosity describing the proliferation of a population of rabbits, it has found widespread use in describing phenomena both in nature and in pure mathematics. Surprisingly, it is connected with the golden ratio, which was proved in relatively recent times to be the limit of the quotient of two successive Fibonacci numbers.
3. *Quiz 4* will occur Thursday May 21.
 - a. That class period will last from 10:45 to 13:15, in TH428 as usual.
 - b. In addition to the quiz, it will feature presentations by seven students.
 - c. The quiz will cover the material mentioned in lectures 21, April 21, through today's, (but not material in student reports). The corresponding lecture files indicate which sections of Struik and Kennedy are covered. The structure of the quiz will be similar to that of the previous ones.
4. *Foundations of natural-number arithmetic*
 - a. Several mathematicians felt the need to pursue the analysis of number deeper than the previously mentioned studies of the real-number system: into the foundations of the *natural* number system: $0, 1, 2, \dots$
 - b. I regard 0 as the first element of the set $\mathbf{N}$ of natural numbers. (Many mathematicians insist on casting 1 in that role, but none who have ever worked in software engineering!)
 - c. There are several routes into this study.
 - i. One is part of the theory of *ordered sets*: it emphasizes the basic properties of the relation $x < y$ between numbers x, y .
 - ii. Another is part of the theory of *cardinality*: comparison of sets by the number of their elements.

- iii. Cantor developed both of those in great detail, particularly for infinite sets, but was not particularly interested in this simple application. We shall ignore those routes.
- d. The route pursued here is a detailed study of the *relationships between basic arithmetic principles*: which depend on which, and which are most fundamental. Grassmann, Dedekind, and Peano pursued this study.
- e. These basic principles include *mathematical induction*:

$$0 \in M \subseteq \mathbf{N} \ \& \ \forall m [m \in M \Rightarrow m + 1 \in M] \Rightarrow M = \mathbf{N}.$$

From now on I'll use the term *recursion* instead of 'mathematical induction'. That is now standard in computer science, and avoids collision with 'induction' in statistical reasoning. (That confusion stems from a mid-1800s book by De Morgan, who introduced both terms in order to contrast the two types of reasoning.) Grassmann investigated the systematic use of recursion to derive other familiar arithmetic laws.

- f. This study is one of *simplification*. You should expect that equivalent intellects given the same simplification problem might attain similar results. That happened. Dedekind carried out his analysis first, but only published it in 1888. Peano worked on the problem during the 1880s, but saw Dedekind's publication only just before his own went to press. Peano acknowledged that his results coincided with Dedekind's, and expressed relief that those confirmed the validity of his own work.
- g. Dedekind and Peano noticed that all concepts used in natural-number arithmetic can be defined in terms of 0, 1, +, and logic, starting as follows:
 - i. $x \leq y \Leftrightarrow (\exists w) w + x = y$
 - ii. $2 = 1 + 1, 3 = 2 + 1, 4 = 3 + 1, \text{ etc.}$
 - iii. $x + 0 = x \ \& \ x + (y + 1) = (x + y) + 1$
 - iv. $x \times 0 = 0 \ \& \ x \times (y + 1) = (x \times y) + x$

The last two lines are *recursive* definitions already known through the work of Grassmann and others. They are recipes for the definition of $x + 1, x + 2, x + 3, \dots$ and $x \times 1, x \times 2, x \times 3, \dots$.

- h. Dedekind and Peano noticed further that after its first appearance '1' occurs in these formulas only in the context '+ 1'. Thus they could regard just 0 and the *successor operation* $x \rightarrow x + 1$ the sole fundamental ideas of natural-number arithmetic, by defining 1 as the successor of 0 and regarding the + signs as applications of the successor operation, not of addition.
- i. Peano phrased his system as follows. Regard as *undefined* a set $\mathbf{N}$, an object 0, and a function $x \rightarrow x + 1$ on $\mathbf{N}$, and let variables $x, y, z, \dots$ range over $\mathbf{N}$. Adopt the following postulates:
 - i. $0 \in \mathbf{N}$;
 - ii. for all $x, x + 1 \in \mathbf{N}$;
 - iii. for all $x, 0 \neq x + 1$;
 - iv. for x, y , if $x + 1 = y + 1$ then $x = y$;
 - v. if $0 \in M \subseteq \mathbf{N}$ and $(\forall x)[x \in M \Rightarrow x + 1 \in M]$, then $M = \mathbf{N}$.

Thus Peano formulated natural-number arithmetic as an axiomatic system in the sense of Aristotle. Postulates (b) and (e) say that $\mathbb{N}$ is closed under the successor operation and it is injective; (f) is the recursion principle.

- j. *Peano's Arithmetices Principia*. [Click here for some pages](#).
 - i. Show the title page and the acknowledgment of Grassmann and Dedekind, page 3 of 20. The book is written in Latin!
 - ii. Note the logical introduction, then the postulates on page 9 of 20.
 - iii. In later versions Peano would relegate the logical postulates to the introductory material.
 - iv. Here is the first explicit use of notation for quantification in a mathematical publication, in the form $\exists x$.
 - v. The recursive definition of $+$ is on page 10 of 20.
 - vi. A recursive proof is presented for the proposition $a + c = b + c \Rightarrow a = b$ (part of proposition 22). The proof notes that the result is a postulate when $c = 1$; and if it holds for some c , then it does for $c + 1$ also, because $a + (c + 1) = b + (c + 1) \Rightarrow (a + c) + 1 = (b + c) + 1$ (by the definition) $\Rightarrow a + c = b + c$ (by a postulate) $\Rightarrow a = b$ (because the result holds for c). This type of proof had been pioneered by Grassmann.
- k. Peano was finally appointed *professor straordinario* in 1890, after Genocchi died.
5. *Göttingen* had attained equality with Berlin as a top mathematics faculty in Germany by the early 1890s, under the leadership of Felix Klein and with major support by the German government.
6. *Klein* had major influence worldwide.
 - a. Particularly in the US:
 - i. He had trained a number of American professors in Germany.
 - ii. He helped set up the faculty at Chicago, which became tops here.
 - b. And in Italy:
 - i. Klein's *Erlanger* program for organizing studies of various types of geometry was a guiding principle for the Italians whose 1880–1910 work I am studying.
 - c. Besides geometry and complex analysis,
 - i. Klein emphasized the development of applied mathematics as a discipline, and
 - ii. wielded major influence on the training of teachers, especially through his books *Elementary Mathematics from an Advanced Standpoint*.
 - d. Klein brought David Hilbert to Göttingen in 1896, and kept him satisfied there.
7. *Hilbert*
 - a. was already famous for revolutionizing algebraic geometry and algebraic number theory.
 - b. He published *Foundations of Geometry* in 1899, now known as the cradle of the modern axiomatic method in mathematics (though the Italians had formulated those ideas earlier).

- c. That led to his formulation of major questions in foundations of mathematics. In particular, *was it possible to prove the consistency of mathematics?*
- d. To attack this problem he advocated regarding mathematical theories as purely formal systems with no associated meaning, and studying proofs simply as sequences of those symbols.
- e. You will hear soon that he *did not* advocate this formalist viewpoint in mathematics in general!
- f. Hilbert gave a famous lecture in 1900 on mathematical problems that, he correctly predicted, would guide mathematics through the next century. Hilbert soon earned general recognition as the world's leading mathematician.
- g. Under the leadership of Klein and Hilbert, Göttingen became the most important mathematical center in the world. It survived hardship during World War I, but even increased its influence during the 1920s.
- h. The Nazis destroyed it in 1933.
- i. But in 1930, Hilbert had made a short radio speech, which incorporated prose he had written much earlier. Some of its most memorable phrases date from the 1900 lecture. Play and read it from my website:

<http://math.sfsu.edu/smith/Documents/HilbertRadio/HilbertRadio.pdf>.