

1. Ms. Lim reported about her first term paper, on Alicia Boole Stott, the daughter of the famous mathematician George Boole. (His name is attached to many concepts in logic and related areas, which stem from a book he published in 1854.) Boole died in 1864, leaving his wife and some children, including the toddler Alicia, nearly without support. Alicia's environment was intellectually very stimulating, but the combination of poverty and customs inhibiting education of women permitted only private education at her mother's side. Alicia studied geometry intensively and followed for many years a line of inquiry about four-dimensional figures. Working with the senior Dutch geometer P. H. Schoute, she published some major results in that field, which helped others to address its problems in visual terms. I believe she also discovered the relationship between the golden rectangles and the regular icosahedron displayed on the mathematical T-shirt modeled in an earlier class meeting by Ms. Howard. After her marriage to Mr. Stott, Alicia deemphasized mathematical activity but returned to it in old age, in conversation with the young mathematician H. S. M. Coxeter, who would become one of the leading geometers of the 1900s. Perhaps as a result of their meetings, he developed a unique ability to visualize four-dimensional geometric relationships, and carried her work to new heights.
2. Mr. DeLeon reported on his first term paper, about the golden ratio φ , which $= a/b$ when $a, b > 0$ and $(a + b)/a = a/b$. Multiplying that equation by the ratio yields $\varphi + 1 = (a + b)/b = \varphi^2$, hence $\varphi = (-1 + \sqrt{5})/2 \approx 1.618$ by the quadratic formula. The Greeks regarded this ratio as especially pleasing when it appears in art forms, and they derived many interesting mathematical properties of φ . Through the centuries, mathematicians extended that work, proving, for example that the ratio of two successive Fibonacci numbers approaches φ as a limit. This result, and the simple nature of that first equation, assures that φ occurs in many natural and scientific contexts. Countless writers have ascribed to many natural phenomena and human artifacts relationships with φ that are not really very close. If something in nature or art is particularly pleasing to the eye, you can generally find some ratio there that is close to φ . Mathematicians know that φ is by no means unique in this regard. But others, dazzled by the mathematical results and the ease of observing their counterparts in nature, too often claim unjustifiable connections. Because of the ease of constructing spectacular illustrations, we now find that probably a majority of the information on φ in popular literature and on the Internet falls in that category. But Mr. DeLeon found and used a very reliable source, the recent book Livio 2005.
3. *Struik, section 8.13: Fourier series and the Riemann integral*
 - a. Recall that a major question in mathematical physics was *what functions have Fourier-series expansions?* When functions are given in that form, many difficult problems involving them become easier.

- b. In fact, you can associate Fourier series to a very large class of functions, as follows, but there is a hitch in the process. Given a function f defined for arguments between -1 and $+1$, consider the series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\pi x + b_n \sin n\pi x),$$

where for each n

$$a_n = \int_{-1}^1 f(x) \cos n\pi x \, dx \quad b_n = \int_{-1}^1 f(x) \sin n\pi x \, dx.$$

This is called the Fourier series of f . If f was originally *defined* as a Fourier series of this sort, then these integral formulas *give back* its coefficients. Further, for many functions f *not* originally defined that way, and for many arguments x , the Fourier series of f *does* converge to f , hence provides a route for the convenient solution of many problems in advanced calculus and differential equations.

- c. Thus the main question has been reformulated as *for what functions f and what arguments x does the Fourier series of f converge to $f(x)$?*
- d. Since the coefficients of the series are integrals, deep analysis of this problem requires deep analysis of the definition of the integral. Cauchy had formulated that definition almost the way we do now. But it was necessary to refine his definition. The form you learned in calculus classes was given by Bernhard RIEMANN (1826–1866) in an 1854 paper on the convergence of Fourier series. [Click here for his original German](#), and [here for an English translation and reference](#). Riemann, one of Gauss's few students, exerted enormous influence on later mathematics.
- e. This definition satisfied mathematicians for a while, and is still used in calculus textbooks. But it is still insufficient for many applications, particularly for probability theory. It has been supplanted by Lebesgue's 1905 integral definition, which is generally introduced only in graduate courses.
- f. There are various other slightly different definitions, some of which are intended as compromises, to get the power of the Lebesgue definition but permit proofs less cumbersome than those needed with that.
4. *Struik, section 8.15: the real number system*
- a. These studies began to involve more and more details about irrational numbers. For example, one of the easiest ways to define a function f that is *not* Riemann integrable on any interval is to set $f(x) = 1$ if x is rational and $f(x) = 0$ otherwise. Irrational numbers are limits of sequences of rational numbers, their decimal approximations. Details about limits are involved, too.

- b. It is straightforward to consider convergence of the sequence of fractions $\frac{2n+1}{3n}$ for $n = 1, 2, 3, \dots$: you notice that it must converge to $\frac{2}{3}$, then show that you can make the difference $\left| \frac{2n+1}{3n} - \frac{2}{3} \right|$ as small as I might require, merely by taking sufficiently large n .
- c. But that requires *prior knowledge* of the limit. What if you don't know that? Consider a familiar example of that situation (pretend you don't know the limit): the sequence of fractions $e_n = \left(1 + \frac{1}{n}\right)^n$ for $n = 1, 2, 3, \dots$. By this time, mathematicians realized that it was probably safe to regard to regard such a sequence as convergent if they could make the difference $|e_m - e_n|$ as small as anyone might require, merely by taking m, n sufficiently large. (That is called *Cauchy's convergence criterion*.) But what does it converge to? Euler had introduced the number e as this limit. What gave him the right to do so? He could approximate it as closely as anyone might require, and evidently it can be manipulated algebraically like other numbers. But how do we know that this will always be the case for numbers *defined* as limits of sequences of fractions and not described otherwise? Nobody had proved that yet.
- d. Only in 1844 was it shown, by Joseph LIOUVILLE (1809–1882), that e is *irrational*. Something fundamental beyond arithmetic of fractions is therefore required to justify our manipulations with it. (This proof uses infinite-series methods covered in calculus classes; I've occasionally presented it there.)
- e. It is possible to follow the route you learned in grade school to justify invention of such numbers. Namely, to define real numbers between 0 and 1 as sequences of digits 0...9: new entities whose manipulation must be justified before we can confidently do decimal arithmetic with them. There are some hitches:
- i. You have to redefine equality to permit $.09999\dots = .10000\dots$, etc. This is not hard to deal with (you learned how in school), just troublesome to describe.
 - ii. You have to describe exactly how to generate the decimal expansion of $a + b$ and ab given those of a and b . This is a mess, and no one, to my knowledge, attempted that in the 1800s, at least in print.
- f. Richard DEDEKIND (1831–1916) was a friend of Riemann and also a student of Gauss. After his 1852 doctorate, he began teaching calculus, working out details of the fundamentals in greater detail than the treatments in published literature. He followed this study even deeper than the derivative and integral, into the fundamental properties of the real number system itself. He invented a definition of real number sufficient for this work. For Dedekind, real numbers are some sort of abstract “names” associated with *cuts* in the

subsets S and L such that all members of S are smaller than all members of L and S has no largest element. Some cuts correspond to rational numbers q : for example $S = \{x \in \mathbf{Q} : x < q\}$ and $L = \{x \in \mathbf{Q} : x \geq q\}$. Others do not, and are called *irrational*: for example $S = \{x \in \mathbf{Q} : x^2 < 2\}$ and $L = \{x \in \mathbf{Q} : x^2 \geq 2\}$.

- g. Dedekind was a teacher in a secondary school. He evidently used his work on the real-number system to introduce calculus there. He waited nearly twenty years to publish. I don't know specifically why, but I suspect that he felt the time was not ripe for it. It makes fundamental and extensive use of *sets*. This was a growing tendency in mathematics, but it took decades for mathematicians in general to become comfortable with that technique. Dedekind published his work in 1872.
- h. What are the mysterious "names" of Dedekind's cuts? In 1903, Bertrand RUSSELL (1872–1970) presented somewhat cleaner definitions. He noted that the names are superfluous. According to Russell, Dedekind could have defined a real number as the cut itself. In 1872, however, mathematicians weren't yet comfortable enough with elementary set theory to permit that. Russell also noted that the sets L in Dedekind's definition are superfluous, if you merely require that each set S be bounded above by some rational.
- i. Meanwhile, back in Berlin, Karl WEIERSTRASS (1815–1897) was developing yet another approach. His lectures attracted students from all over Europe, and they spread the word. The details are no longer familiar. But Weierstrass was responsible for the *arithmetization* of analysis: the ε, δ method of dealing with limits, that you learn in classes now. This was disseminated not through published papers so much, but through his students' later lectures and use of the method in their papers.
- j. Another German who worked in the same vein was Hermann GRASSMANN (1809–1877), from the Berlin school but largely self-taught. His work extended more into foundations of natural-number arithmetic and of vector analysis. He wrote very obscurely, and thus had little influence in mathematics until Peano adapted some of his techniques. Grassmann worked in several other areas, and had very major influence in Sanskrit philology!
- k. It's worth noting that Grassmann, Weierstrass, and Dedekind spent large parts of their academic careers teaching in schools below university level.
- l. Each of Dedekind and Weierstrass showed that his system of real numbers enjoyed the *continuity property*: for every cut S, L in the set $\mathbf{R}$ of *real* numbers (in Dedekind's sense) the set L has a smallest element r . That is, $S = \{x \in \mathbf{R} : x < r\}$ and $L = \{x \in \mathbf{R} : x \geq r\}$. This is virtually the same as the *least-upper-bound principle*: every nonempty set of real numbers that is bounded above has a least upper bound. You can derive either of these from the other quite easily. Various mathematicians had already used these properties of the real number system to prove results fundamental to calculus. But previously, they had not analyzed the real number system sufficiently to derive these properties.

- m. The properties are fundamental to the underpinnings of calculus. One or the other provides the key step in showing that
 - i. each Cauchy sequence converges,
 - ii. continuous functions on finite nonempty closed intervals
 - (1) have maximum values,
 - (2) satisfy the intermediate-value theorem,
 - (3) and have definite integrals.
- n. You'll find the continuity or least-upper-bound principle mentioned in many precalculus calculus and most calculus texts, and every beginning real-analysis course is based on it.