

1. *Assignment.*
 - a. Kennedy, chapters 3–4.
 - b. Here's the plan for Struik, chapter 8.
 - i. 8.1–8.4 will be finished today.
 - ii. 8.5 —skip it.
 - iii. 8.6 was covered already in lecture 24.
 - iv. 8.7 will start today.
 - v. 8.8 —skip it.
 - vi. 8.9 —you're responsible for the material on Abel and Galois and solubility of polynomial equations with degree > 4 , but I won't cover it in lecture. (next), 8.15, 8.26.
 - vii. 8.10–11 —skip them.
 - viii. 8.12 was covered already in lecture 24.
 - ix. 8.13–15 will be covered.
 - x. 8.16–17 —skip them.
 - xi. 8.18 is to be decided later.
 - xii. 8.19–25 —skip them.
 - xiii. 8.26–27 will be covered.
2. Ms Vergara described her first term paper, on Sophie Germain. Germain experienced the French revolution as a child in a middle-class merchant family. She was self- or privately taught in mathematics, using her father's books. As a woman, she was not allowed to enroll in any French university, but arranged to have notes taken, and to present her work under a male pseudonym. Lagrange discovered this, and served afterward as her mentor. Sophie overcame many other obstacles to achieve notable research results in the theory of elasticity and in number theory.
3. Mr Koehler described his first term paper, on fractals. The first example was constructed by Peano in the early 1890s as an example to show that counter to the prevailing opinion, a continuous curve, commonly regarded as one-dimensional, could indeed pass through *every* point of a two-dimensional region in the plane. A year later, Hilbert simplified it. Such examples received only sporadic attention until about thirty ago, when Benoit Mandelbrot studied them intensively, using methods invented by Hausdorff about a century ago to clarify the notion of dimension and permit dimensions strictly between one and two, for example. Sets with such dimension are called *fractals*. His work has created a large new area of mathematics, with many applications.
4. *Struik, sections 8.1–8.4: Gauss and Legendre, continued*
 - a. *Numerical integration.* Gauss's "practical" work in astronomy required very precise numerical approximation of integrals. He invented an optimal variant of the Newton-Cotes formulas that include the trapezoid and Simpson rules. I learned it in that course senior year, and it was the only piece of really advanced mathematics that I used while working as a mathematician for the

Navy for about 5 years in the 1960s. It is formulated in terms of roots of Legendre polynomials. I learned recently that the designers of *Mathematica* chose a further elaboration, the Gauss–Kronrod method, for standard use in that software.

- b. *Differential geometry*. While working on problems in geodesy, Gauss developed the theory of surface curvature. That was a capstone of my undergraduate calculus classes, and was featured again in the differential geometry course I took in my first graduate year at SFSU.
 - c. *Noneuclidean geometry*. Gauss developed independently part of what is now called noneuclidean geometry, but never published it. Its only effect on later mathematicians was to enhance their regard for the work of those who did publish on that subject. This was never in any course I took, although my PhD research was on that subject.
5. *Struik, section 8.7: Cauchy*
- a. Some of Cauchy's work complements Gauss's, so I'll detail Cauchy's similarly. But Cauchy's main import for this course is work on foundations of calculus and on differential equations, both of which lead straight to Peano.
 - b. Cauchy's personal life was disrupted by war and political strife. He was born in 1789 on the eve of the French revolution, and his Royalist family were refugees during that time. He entered l'École Polytechnique in 1805—during Napoleon's reign—but transferred to the school of bridges and highways, graduated as an engineer in 1809, worked a while for the Navy, then became a professor at the Polytechnique and other Paris schools from 1816 on. He became famous for his research and his textbooks. Until 1830 he flourished under the restored Bourbon monarchy.
 - c. [Click here](#) for a biographical sketch and some portraits of Cauchy gathered by St. Andrews. Notice that we witness here the beginning of photography!
 - d. Cauchy evidently felt endangered by the 1830 revolution that installed the "citizen king" Louis Philippe (still from the Bourbon family) and brought many democratic reforms. Cauchy fled to Switzerland, then to Turin (at that time in the Kingdom of Sardinia), then to Prague (then in Austria). There he worked as a tutor to children of the deposed branch of the Bourbons.
 - e. Turin mathematicians of Peano's time evidently were very proud of their city's earlier involvement with Cauchy.
 - f. Cauchy returned to France in 1838 but was barred from some of his positions. Nevertheless he continued prolific research activity. Cauchy seems to have avoided deep involvement with the revolution of 1848 that ended the Bourbon monarchy, made Louis Napoleon (Bonaparte) President of the Second Republic, and transformed him into Emperor Napoleon III in 1852. Cauchy died in 1857.
 - g. Cauchy's mathematics that was most closely related to Gauss's was his formulation of the basic theorems of *calculus for the complex number system*, particularly the Cauchy–Riemann equations, which were known long before Cauchy and Riemann, and Cauchy's integral theorem and integral formula.

Because those are not directly related to Peano's work, I'm not going to cover that now.

- h. Cauchy contributed very many results in various areas of applied mathematics (remember, he was an engineer), which I will also pass over because they don't lead to Peano.
- i. A *first-order ordinary differential equation initial value problem* (ODEIVP) is formulated as follows. Given a real-valued function F of two real variables x and y , and given an argument x_0 and a value y_0 , find a differentiable real-valued function f such that

$$f(x_0) = y_0 \quad f'(x) = F(x, f(x)) \quad \text{for all } x \text{ of interest.}$$

For a century mathematicians had been solving individual ODEIVPs, particularly those of great importance in physics. Cauchy was able to prove that there always exists a solution, provided F satisfies certain differentiability conditions. Peano would take up that argument and push it farther, eliminating some of the conditions on F . This result is often the first one stated (but not proved) in elementary ODE courses. It assures that the quest for a solution using calculus is not hopeless.

- j. Cauchy's calculus textbooks became famous for their precision and rigor.
 - i. In higher-dimensional calculus he evidently fostered the use of the inequality $|\alpha \cdot \beta| \leq \|\alpha\| \|\beta\|$ involving the dot product and norms of vectors α and β . This is called the Cauchy–Bunyakovski–Schwartz inequality: the order of the names can vary with your politics. Vector terminology and notation would not become standard until the early 1900s—Peano was involved, but not necessarily instrumental, in that, too.
 - ii. Cauchy formulated the derivative as a limit the way we do now, and the definite integral almost as we do now.
 - iii. He formulated and proved the mean-value theorem, which is the primary tool for error analysis in approximation theory.
 - iv. But I believe a complete proof of that result requires the theorem that a continuous function on a closed bounded interval assumes a maximum value. I think Cauchy did not have that detail of the proof yet. This, and the equally critical intermediate-value theorem were provided later by Bolzano, Weierstrass, and others.
- 6. After Cauchy, many questions remained about whether different limiting processes commute:
 - a. Is the derivative of the limit of a sequence of functions the limit of the sequence of their derivatives?
 - b. Is the integral of the limit of a sequence of functions the limit of the sequence of their integrals?
 - c. Is the derivative of the integral of a function with respect to one of two variables the integral of its partial derivative with respect to the other?
 - d. Similar questions involving infinite sums.
 - e. When is it valid to change the order of partial differentiation?

During the mid-1800s, mathematicians straightened out many of the problems of this sort. But for many years slapdash reasoning continued to lead to episodes of unreliable calculus. Some problems of this sort remained for Peano and others to tackle as late as the 1890s.