

1. *Assignment.* I'm starting a perhaps confusing practice of covering parts of Struik, chapter 8, but not in order. The first sections I'll cover are 8.6 and 8.12. You'll have to use these notes to keep track of the sequence.
2. Mr. Weddick spoke about his first term paper, on Dirk Struik. He emphasized Struik's early commitment to Marxist principles, and Struik's quandary, whether to be a mathematical Marxist or a Marxist mathematician. Struik chose the latter. Struik's work had major influence on later histories of mathematics and science, particular in their featuring of the social and political context of scientific achievement. Struik was the center of major controversy in the Cold-War McCarthy Era, when he was suspended from teaching at MIT for some time because of his political beliefs. The eventually favorable outcome of that dispute was a victory for liberal academia over the McCarthyites.
3. Ms. Sheldon spoke about her first term paper, on Maria Agnesi. She stressed Agnesi's eighteenth-century upbringing in a wealthy Milan family that hosted frequent salons, gatherings of highly literate and articulate people to discuss the literary, political, and scientific questions of the day. A multilingual prodigy, Agnesi spoke at length to these salons as early as age 9, particularly urging education of girls. Following her father's interest, she studied mathematics deeply, and in her twenties published a four-volume study of the mathematical research of the time. She tended to the education of twenty(!) siblings, entered a convent in her thirties, supervised the administration of its services to the poor, and died there at age 80.
4. *Peano biography, chapter 2.*
 - a. Peano is entering the university of Turin. [Click here](#) for an 1890 picture of its location between Via Verdi and Via Po. (The tower, called the Mole Antoniella, was under construction when Peano was a student. It was first intended as a synagogue, but later sold to the city. It's now a museum of cinema.) [Click here](#) for a 2008 picture of the interior courtyard of the main building at that time. It's now an administration building.
 - b. Terms
 - i. *collegio* (residence: Peano was needy)
 - ii. *solemn inauguration* (convocation)
 - c. Numbers of students:
 - i. 1334 in the university (large for Italy then)
 - ii. 103 in first-year mathematics
 - iii. 1 in third-year mathematics
 - d. Classes:
 - i. Freshman
 - (1) algebra, analytic geometry with D'Ovidio
 - (2) projective geometry with Bruno
 - (3) design, chemistry
 - ii. Sophomore

- (1) calculus with Genocchi
 - (2) descriptive geometry with Bruno
 - (3) physics with Basso
 - (4) design, geology
- iii. Junior
 - (1) higher geometry with D'Ovidio
 - (2) analysis with Fàa di Bruno
 - (3) mechanics with Erba
- iv. Senior
 - (1) higher mechanics with Siacci
 - (2) physics with Basso
 - (3) higher geometry with D'Ovidio
- e. You'll find in Smith & Marchisotto 2007, section 1.3, biosketches of all these named professors except Fàa di Bruno, whom you can read about online in English. (He was beatified some years ago.) D'Ovidio and Genocchi probably had the most influence on Peano.
- f. Peano won awards—fee waivers—consistently, and was one of the two who passed the fourth-year examinations in mathematics (four attempted).
- g. As I mentioned earlier, I don't know whether he wrote a dissertation.
- 5. *Peano biography, chapter 3*
 - a. In 1880, immediately after earning the doctorate, Peano obtained a position at the university, as assistant to D'Ovidio, his professor in geometry courses.
 - b. D'Ovidio presented Peano's first research paper, on algebraic geometry, to the Turin Academy of Sciences that same year. [Click here](#) for a picture of D'Ovidio, [here](#) for the exterior and interior of the Academy, and [here](#) for and a map of downtown Turin. The Academy is in the building now called the Museo Egizio (Egyptian Museum). I suspect its interior looked the same in Peano's time. A nearby street is named for Lagrange. The university is on Via Po, near the Academy, on the same type of street, and Peano took lodgings on Via Po as well.
 - c. [Click here](#) for the paper, Peano 1881. What's it about?
 - i. An (m,n) -correspondence in the plane is a relation between points x,y defined by a polynomial equation $P(x,y) = 0$, such that in general, to each x correspond n values of y , and to each y , m values of x . (Explaining the phrase *in general* would involve explaining the use of complex coordinates and multiplicities of roots of polynomials.)
 - ii. Peano showed that for $m,n \leq 2$ there is a ruler-and-compass algorithm that, given x , will construct all corresponding y , or given y , all corresponding x . He indicated that this would not be true for larger m,n .
 - d. Eventually, Peano was transferred to become assistant to Angelo Genocchi, his former calculus professor. Soon, Genocchi fell ill, and Peano took over his entire teaching load. Peano revised Genocchi's notes and published them as the book Genocchi 1884. Genocchi claimed that it was really Peano's book, not his, but that authorship actually gained recognition for Peano. Peano

incorporated there a number of the delicate arguments that we now use for various theorems involving limits. Earlier texts had handled such matters informally and sometimes incorrectly.

- e. [Click here](#) for a picture of a bust of Genocchi, in the Academy of Sciences.
- f. [Click here](#) for a photograph of the dashing young Peano.
- g. [Click here](#) for the title, contents, and preface pages of the book. Note Genocchi's homage to Peano, and the highlighted paragraph:

... ed infine credetti utile di far ivi notare alcune inesattezze in proposizioni e dimostrazioni, che sono quasi stereotipate in un gran numero di trattati, e che si riproducono ancora in opere recentissime, benchè in gran parte la loro inesattezza fosse già rilevata da varii autori, anche da anni.

Here it is in English:

... and finally I thought it useful to take note here of some inexactnesses in propositions and demonstrations, that are found stereotypically in a great number of treatises, and which are reproduced again in recent works, even though to a large extent their inexactness had already been mentioned by various authors over the years.

Thus, Peano was already taking on the mathematical world, to reform the presentation of mathematics, to render it in a much more precise manner than had been customary.

- h. [Click here](#) for two highlighted paragraphs from its section 6, rendered in English as follows:

6. In the questions that we shall consider, some quantities can appear, to which, it is supposed, are attributed determined and fixed values, and which are called *constants*; and others that, it is supposed, can assume diverse values, and which are called *variables*. Among the variables there are some to which we can attribute diverse values successively at will, and which are called *independent variables*; and others whose values depend on the values given to the first ones, and which are called *dependent variables* or *functions of the first ones*.

We will consider from the onset functions of a single independent variable, and we shall say that a function y of x is given in an interval (a, b) , if to each value of x that falls between a and b corresponds a unique and determined value of y — whatever be the means of determining it.

Here Peano noted that he would *not* use the notion of function as it had been described by Leibniz, Euler, and Lagrange. They had insisted that functions always be given by specific formulas, or analytic expressions. Instead, Peano would employ the mid-1800s notion introduced by Dirichlet, who considered functions merely as correspondences between arguments and values, not necessarily given by formulas.

- 6. *The definition of "function".*
 - a. I'll use Genocchi 1884 and Struik, sections 8.6, 12 to trace this evolution.
 - b. [Click here](#) for Peano's highlighted note to the reader about section 6, quoted in item 4.h of the present outline.
 - c. The cited paper of Dirichlet is about Fourier series. We need a digression.
- 7. *Fourier series.*
 - a. Struik mentioned Fourier series in passing. I want to give more detail because that method figures importantly in what follows. My discussion is based on

Boyce and DiPrima 1997, chapter 10, which I think follows early-19th-century ideas closely.

- b. Consider a homogeneous heat-conducting rod, insulated from its environment, with cross sections of uniform area A perpendicular to its axis, the unit interval. Let

$u(x, t)$ = temperature at time t at points with axial coordinate x .

Think of u as the heat *density* at x, t and think of heat as a fluid. Thus heat is like water, but it has energy rather than mass. Consider what happens over a time interval Δt to a slab between two cross sections with coordinates x and $x + \Delta x$. By Newton's law (experimentally verified somehow) the amounts of heat crossing the sections in the positive direction are

$$-k \frac{\partial u}{\partial x} \Big|_{x,t} A \Delta t, \quad -k \frac{\partial u}{\partial x} \Big|_{x+\Delta x,t} A \Delta t,$$

respectively; k is a proportionality constant called the *conductivity*. Thus the net amount of heat entering the slab is

$$k \left(\frac{\partial u}{\partial x} \Big|_{x+\Delta x,t} - \frac{\partial u}{\partial x} \Big|_{x,t} \right) A \Delta t = k \frac{\partial^2 u}{\partial x^2} \Big|_{x+\zeta \Delta x,t} A \Delta x \Delta t, \quad (1)$$

for some ζ in the unit interval, by the mean-value theorem. We must make various continuity assumptions for that argument to work. Further such assumptions imply that the amounts of heat in the slab at times t and $t + \Delta t$ are

$$u(x + \eta \Delta x, t) A \Delta x \quad u(x + \theta \Delta x, t + \Delta t) A \Delta x,$$

respectively, where η and θ lie in the unit interval, because the product $A \Delta x$ is the slab's volume. Thus the amount of heat entering the slab is also

$$\frac{u(x + \eta \Delta x, t + \Delta t) - u(x + \theta \Delta x, t)}{\Delta t} A \Delta x \Delta t. \quad (2)$$

Equations (1) and (2) imply

$$k \frac{\partial^2 u}{\partial x^2} \Big|_{x+\zeta \Delta x,t} = \frac{u(x + \zeta \Delta x, t + \Delta t) - u(x + \theta \Delta x, t)}{\Delta t}.$$

Applying further continuity assumptions, as required, to take limits as $\Delta x, \Delta t \rightarrow 0$ yields the *heat equation*

$$k \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}.$$

- c. Given a function f on the unit interval, such that $f(0) = 0 = f(1)$, is it possible to find a function u that satisfies the heat equation and the auxiliary conditions

$$\begin{aligned} u(0,t) &= 0 = u(1,t) \text{ for all } t \\ u(x,0) &= f(x) \text{ for all } x \text{ in the unit interval?} \end{aligned}$$

- d. If this problem can be solved, so can many problems with other boundary conditions, by changing variables.
- e. Fourier observed around 1800 that if

$$f(x) = \sum_{n=1}^{\infty} b_n \sin n\pi x$$

for all x in the unit interval, where the b_n are constants, then this is a solution of the problem:

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2 \pi^2 kt} \sin n\pi x$$

- f. Thus, solution of the heat equation problem is possible if f has an expansion of the form suggested, which is called a *Fourier series*. This brings up the very major question, *what functions have Fourier series expansions?*
- g. Verifying that Fourier's solution actually satisfies the heat equation requires the convergence of that series and its term-by-term differentiability. That, and the validity and use of the mean-value theorem, were not fully established until later in the 1800s.
- h. [Click here](#) for a biographical sketch of Joseph Fourier (1768–1830). Please realize that this Fourier is *not the same* person as the Fourier famous for fostering utopian communities in this country.
8. *What is a function?*
- a. We saw earlier that until the 1800s, mathematicians had assumed that calculus was the study of functions given by algebraic formulas of various sorts. Genocchi 1884, edited by Peano, included quotations from Leibniz and Euler to this effect.
- b. During the 1700s and early 1800s mathematicians began applying calculus methods to functions that do not necessarily have this form. Some of those have formulas of types not considered before, but in some cases there is no obvious reason why there should be any formula at all. For example, consider the function f in the heat conduction problem discussed in the previous entry of this outline.
- c. This led to the notion of real function f that we use now: an assignment of unique values $f(x)$ to all arguments x in some set D of real numbers, called its *domain*.
- d. Struik mentioned that this originated with Peter Gustav Lejeune Dirichlet (1895–1859). [Click here](#) for a biographical sketch. Caution: some sources regard “Lejeune” as a forename, and alphabetize him under “D”; others regard it as part of the surname, and alphabetize him under “L”.