

1. *Assignment.* Read Struik, chapter 4.
2. From now on, *email* me questions about your term papers. You automatically get a copy for your records, and hardcopy tends to get lost.
3. *Area, continued*
 - a. Ms Fick will derive the formula for the area of a triangle at the next meeting.
 - b. Mr. Banth derived those for the areas of a trapezoid and a parallelogram.
 - c. The method used for their proofs is called *dissection*.
4. *Pythagorean theorem*
 - a. Mr. Lam derived the equation $a^2 + b^2 = c^2$ relating the legs a, b and hypotenuse c of a right triangle, using a dissection argument.
 - b. I derived it another way, seemingly independent of area ideas, as follows.
 - i. Develop a theory of similar triangles.
 - ii. The foot F of the perpendicular from the vertex C of the right angle of right $\triangle ABC$ divides its hypotenuse into two segments, of lengths d, e , adjacent to the legs opposite A and B , which have lengths a, b . Use that theory to show that $\triangle ABC \sim \triangle CBF \sim \triangle AFC$.
 - iii. Let c be the length of the hypotenuse, so that $d/a = a/c$ and $e/b = b/c$ (mean proportionals!).
 - iv. These equations imply $a^2 + b^2 = cd + ce = c(d + e) = c^2$.
5. *Irrationality of $\sqrt{2}$*
 - a. Applying Pythagoras' theorem to one of the right triangles formed by two edges of a unit square and a diagonal, of length d , one gets $2 = 1^2 + 1^2 = d^2$. That is, $d = \sqrt{2}$.
 - b. If d were a rational number, the ratio m/n of two whole numbers with no common factor, then m, n could not both be even, and $m^2 = 2n^2$.
 - c. Then for some whole numbers p, q one of the following is true:
 - i. $m = 2p$ and $n = 2q + 1$, or
 - ii. $m = 2p + 1$ and $n = 2q$, or
 - iii. $m = 2p + 1$ and $n = 2q + 1$.
 - d. Any one of these pairs of equations, with $m^2 = 2n^2$, leads to an equation of which one side is even and the other odd: a contradiction.
 - e. Thus d is *irrational*.
 - f. The Greeks learned that in Pythagorean times. That's why they couldn't use the numbers they knew, the *rationals*, for their geometric calculations.
6. *Area and similarity*
 - a. Euclidean geometry can be presented rigorously, defining the real number system, then areas, geometrically. No specific area postulates are required. Even without a completely rigorous theory of the real numbers (which was not achieved until the late 1800s), definition of area seems straightforward: just *define* the area of a triangle to be half its base times its altitude, and

define the area of a polygonal region to be the sums of the areas of its constituent triangular regions. Two questions arise.

- i. Which edge of a triangle is its base? To show that the area given by the formula just mentioned does not depend on the choice of base, you need a theory of similarity that applies to all triangles, not just those whose edges can be measured by rational numbers. The Greeks did work that out. One of the major mathematical achievements of all time, comparable to the theory of limits used in our analysis courses, it's presented in Euclid. Just who originated it is still a matter of serious discussion.
 - ii. Is the sum of the areas of the constituent triangular regions independent of the choice of those regions? Yes, but that was proved only in the early 1800s, by Farkas Bolyai, father of one of the originators of non-Euclidean geometry. The proof is tedious but not difficult.
 - b. One modern approach to Euclidean geometry, followed in Smith 2000, uses area postulates like those mentioned here not just to derive the familiar area formulas, but also to develop the full theory of similarity. (It bypasses the construction of the real number system by assuming special postulates.) I learned this approach from the marvelous book Moise [1963] 1990, which is the basis of many contemporary high-school geometry texts.
 - c. Thus my earlier proof of the Pythagorean theorem using similarity theory only *seemed* independent of area considerations. Similarity and area theory are in a way equivalent.
7. *Volumes of polyhedra*
- a. Euclid didn't distinguish the volume of a figure from the figure itself, since he had no concept of volume as a real number. Just as with areas, that makes discussion awkward. To compute volumes he used the dissection method. But he found that inadequate, as we shall see.
 - b. Here's a rough description of his *dissection method*, phrased in more modern terms of numbers.
 - i. *Tetrahedral* regions are defined analogously to triangular plane regions.
 - ii. *Polyhedral* regions, analogous to polygonal regions, are built from tetrahedral ones that intersect only along whole faces.
 - iii. [Volumes of] polyhedral regions that intersect only along common faces can be *added*, just like [areas of] polygonal regions.
 - iv. Congruent polyhedral regions are *equal* [in volume].
 - c. I haven't prepared material on the beginnings of his theory, but from the list of propositions in Euclid's books XI and XII, I note his development to this point: [the volume of] a prism is proportional to its altitude and its base [area]. Thus prisms with congruent altitudes and bases equal [in area] are equal [in volume]. This means that he could carry out reasoning equivalent to our formula, base area times altitude, for the volume of a prism. Since boxes are prisms, that would yield our formula for the volume of a box.