

1. *Volunteers wanted* to present brief term-paper research progress reports two weeks from today, Tuesday 10 March. I am speaking to a colloquium at Simpson College in Redding that Monday evening, and have to drive back Tuesday morning. This activity can take place without me, in case I am delayed.
2. *Student paper contest*. The Mathematical Association of America sponsors a student history paper contest, whose requirements are close to those of your first paper for this course. [Click here for their flyer](#). If you're starting a project that might be competitive in such a contest, let me know. Their deadline is after ours, but nevertheless, you'd want to push hard and early if you'd like to compete—that kind of goal always begs for extra-intense last-minute effort, and you'd need to have time for it.
3. *Social history*
  - a. A student inquired about investigating the relationship of mathematics and industrialization. That's an intriguing subject, about which I've been curious for a long time, but, sadly, never spent much time on. I just read Struik 1986, which has many references, some of which I'm following myself. Some are about science in general and industrialization. It might be best to find some particular example, then concentrate on that.
  - b. I mentioned the mathematics at the 1900 Paris world's fair as a possible paper topic. Some of that will be featured in the last lecture of this course. Now there is a temporary exhibit at the Legion of Honor, entitled Fabergé, of the flamboyant jewelry and art objects of that time. Fabergé was jeweler to the Russian court. He and his competitors, the American Tiffany pair, father and son, and the French Lalique, all had big galleries at the fair. The work they displayed is featured at the Legion exhibit, with photos of the original surroundings. This is a good way to get an idea of the social milieu in which mathematics was developing at that time.
4. *Existence of a parallel, continued*
  - a. *Perpendiculars to a line at distinct points  $A, B$  must be parallel*. For if they met at a point  $C$ , then  $\triangle ABC$  would violate the exterior angle inequality.
  - b. *Dropping a perpendicular* (Euclid I.12). Given a point  $P$  not on a line  $g$ , construct the foot of a perpendicular from  $P$  to  $g$  as follows.
    - i. Let  $A$  be any point on  $g$ . If  $AP$  is perpendicular to  $g$ , we are done.
    - ii. Otherwise, construct the circle  $\mathcal{C}$  through  $A$  centered at  $P$ ; it must intersect  $g$  also at some point  $B \neq A$ . (This has to be justified by one of the additional postulates mentioned earlier.)
    - iii. By the isosceles triangle theorem,  $\angle A \cong \angle B$ .
    - iv. Construct the midpoint  $M$  of  $AB$ .
    - v. By SAS,  $\triangle AMP \cong \triangle BMP$ , and therefore  $\angle AMP \cong \angle BMP$ .
    - vi. That is,  $PM \perp g$ , as required.

- c. *Erecting a perpendicular* (Euclid I.11). Given a point  $M$  on a line  $g$ , construct a perpendicular to  $g$  at  $M$  as follows.
    - i. Let  $A$  any other point on  $g$  and extend  $AM$  past  $M$  to a point  $B$  such that  $AM = MB$ .
    - ii. Construct an equilateral triangle  $\triangle ABP$  with base  $AB$ .
    - iii. Then  $\angle MAP \cong \angle MBP$  by the isosceles triangle theorem, and therefore  $\triangle AMP \cong \triangle BMP$ .
    - iv. It follows that  $\angle AMP \cong \angle BMP$ , so that  $PM \perp g$ , as required.
  - d. The proof of the existence of a parallel to a given line through a point not on it is now complete.
  - e. The proof did not use the parallel postulate. Moreover, I was careful to use only the SAS congruence principle, never the SSS or ASA principles.
5. *Angle construction and congruence principles*
- a. To demonstrate the equivalence of Euclid's and Playfair's versions of the parallel postulate, I need one more result, which is independent of them: construction, at a given location, of an angle congruent to a given angle (Euclid I.23). Given  $\angle ABC$  and distinct points  $D, E$ , to find a point  $F$  on a given side of line  $DE$  such that  $\angle ABC \cong \angle DEF$ , proceed as follows.
    - i. Construct the foot  $C'$  of the perpendicular dropped from  $A$  to line  $BC$ .
    - ii. Construct  $D'$  on line  $DE$  so that  $D'E = C'B$ .
    - iii. Erect a perpendicular  $g$  to line  $DE$  at  $D'$ .
    - iv. On the given side of  $DE$  construct  $F$  on  $g$ , so that  $D'F = AC'$ .
    - v. By SAS,  $\triangle ABC' \cong \triangle FED'$ , and therefore  $\angle DEF = \angle D'EF \cong \angle C'BA = \angle ABC$ , as required.
  - b. The usual demonstration that the usual method for constructing an angle actually works uses the SSS congruence principle. But I have avoided that, because Euclid's proof of SSS is faulty in the same way his proof of SAS was. I found the proof I just gave, which uses only SAS, in Mueller 1981, 23. That book is the best treatment of these matters, at least in English.
  - c. There is a correct proof of SSS, based on SAS, which is presented in many modern approaches to Euclidian geometry, such as Smith 2000. I wanted to save time here by avoiding even that. Mathematicians had discovered by the late 1800s that the SAS principle implies all the other congruence principles, so that it's only necessary to assume SAS as a postulate. I don't know whether it's possible to assume any one of the others in its place. (That's a reasonable MA project.)
6. *Euclid implies Playfair*
- a. Now assume Euclid's parallel postulate: if two lines  $g, h$  intersect a line  $k$  at points  $A, B$ , and  $E, F$  are points of  $g, h$  on the same side of  $k$ , and  $m\angle ABF + m\angle BAE < \text{two right angles}$ , then  $g, h$  intersect on that same side of  $k$ .
  - b. Playfair's proposition says that given a point  $P$  and a line  $g$  not through it, there's *at most* one line through  $P$  parallel to  $g$ . It is derived from Euclid's postulate as follows:

- i. Following lecture 7, drop  $h \perp g$  from  $P$ , and erect  $k \perp h$  at  $P$ , so that  $k \parallel h$ .
  - ii. Suppose  $l$  were any other line through  $P$ , so that it's not  $\perp g$ .
  - iii. If  $l$  made more than a right angle with  $h$  on one side, it would make less than a right angle on the other. Therefore, on one side or the other, it makes less than a right angle.
  - iv. On that side,  $g$  and  $l$  make angles with  $h$  whose sum is less than two right angles. By Euclid's postulate,  $l$  cannot be parallel to  $g$ . That's what was to be shown.
- 7. *Playfair implies Euclid*
  - a. Now instead of assuming Euclid's form of the parallel postulate, assume Playfair's: given a point  $P$  and a line  $g$  not through it, there's *at most* one line through  $P$  and parallel to  $g$ .
  - b. I can then prove Euclid's form as follows.
    - i. Suppose two lines  $g, h$  intersect a line  $k$  at points  $A, B$ ; and  $E, F$  are points of  $g, h$  on the same side of  $k$ ; and  $m\angle ABF + m\angle BAE < \text{two right angles}$ .
    - ii. Use the angle-construction principle described earlier (which is independent of the parallel postulate) to construct  $G$  on the same side of line  $AB$  as  $E, F$  so that  $m\angle ABG = \text{two right angles} - m\angle EAB$ , and therefore line  $BG \neq h$ .
    - iii. If line  $BG$  intersected  $g$  at some point  $H$ , then  $\triangle BHA$  would violate the exterior angle inequality; therefore,  $BG \parallel g$ .
    - iv. By Playfair's proposition,  $h$  must intersect  $g$  at some point  $J$ .
    - v. If  $J$  were on the side of  $k$  opposite  $E, F$ , then  $k$  would have to intersect segment  $FJ$ , which is impossible.
    - vi. The proof of Euclid's postulate from Playfair's is complete.
- 8. *The Greeks regarded geometry and arithmetic as separate.*
  - a. Geometry dealt with segments, triangular regions, etc.
  - b. Arithmetic considered the integers and rational numbers but nothing resembling our real numbers.
  - c. This created a problem because they knew that no rational number would measure the length of the diagonal of a unit square, or of many other useful segments.
  - d. Rather than regarding segments and regions as having numerical lengths and areas appropriate for calculation, they had to manipulate the segments and regions themselves. They would add and subtract segments, add and subtract triangular regions, etc.
  - e. They discovered a rigorous way of doing that, but it's very tedious. Part of their theory involved elaborate approximation techniques rather like those used in our real analysis courses.
  - f. But some parts of it were much simpler. We'll consider some of those results about area next, but we'll regard segments and regions as having *numerical*

lengths and areas, which we can manipulate arithmetically as we learned in school. (Our arithmetic techniques were developed much later.)

9. *Some areas*

- a. A triangle together with its interior constitute a *triangular region*. The union of some triangular regions that intersect at most along entire edges is called a *polygonal region*  $R$ . Its *interior* is the set of centers of circles that lie entirely inside  $R$ ; the other points of  $R$  constitute its *boundary*. With some diligence it can be shown that the boundary is the union of the edges of the triangles that are not those intersections, and that the union of two polygonal regions that intersect at most along their boundaries is also a polygonal region.
- b. I'll assume four postulates about area:
  - i. All polygonal regions have areas.
  - ii. Congruent triangular regions have equal areas.
  - iii. If region  $A$  is the union of regions  $B, C$  that intersect at most along their boundaries, then the area of  $A$  is the sum of those of  $B, C$ .
  - iv. A square region with edge of length  $s$  has area  $s^2$ .
- c. From these and algebra I easily derived the familiar formula for the area of a rectangle.