

1. Quiz 1 was debriefed via hardcopy.
2. Ms Johnson summarized two reviews of Kennedy's biography of Peano, by Ivor Grattan-Guinness and Joan Richards. Both are distinguished historians of mathematics.
3. *Euclid's proposition 5, continued.* If $AB = AC$ in $\triangle ABC$, then $\angle B = \angle C$. I gave Pappus' short proof during last lecture. Ms Howard presented Euclid's much more complicated proof. This is an example of a common phenomenon: proofs of mathematical results gradually become simpler as mathematicians understand them better and better.
 - a. Use previous propositions to extend AB and AC past B and C to points F and G so that $AF = AG$, and therefore $BF = BG$.
 - i. Then $\triangle ABG \cong \triangle ACF$ by SAS, and therefore $FC = BG$, $\angle CFA \cong \angle BGA$, and $\angle ABG \cong \angle ACF$.
 - ii. By the equation $BC = CB$, the first two results in (ii) and SAS, $\triangle BCF \cong \triangle CBG$, and therefore $\angle CBG \cong \angle BCF$.
 - iii. By the last results in (ii) and (iii), $m\angle B = m\angle ABG - m\angle CBG = m\angle ACF - m\angle BCF = m\angle C$, as desired.
 - b. This proposition is fondly called (*pons asinorum*—the bridge of asses), perhaps because it marks reluctant students' transition to understanding, or because it looks like bridges often built because an ass sometimes won't cross even a narrow stream, or because that reluctance causes its driver to find a detour, like the one around the main triangle in the proof.
 - c. Pappus's proof required that triangles be regarded as *ordered triples* of noncollinear points, not just *sets* of three noncollinear points. While that may seem sophisticated, it's not, because you really can't set up any theory of triangle congruence without regarding triangles that way. That many teachers and students stumble at the distinction is probably due to habitual use of imprecise language from an early age. Pappus didn't stumble. (Some very elementary aspects of programming are also unintelligible unless you use that kind of language precisely.)
4. *Euclidean constructions*
 - a. Some scholars claim that Euclid's goal was not an axiomatic presentation in the sense of Aristotle, but a manual of methods for carrying out geometric constructions. I rather like that idea, since it frees Euclid from many criticisms of gaps in his postulate set.
 - b. Possible paper topic: Euclidean constructions.
 - i. Euclid himself showed that with straightedge and collapsing compass you can do anything that can be done with straightedge and noncollapsing compass.

- ii. There are surprising results about what you can do with just straight-edge, or just compass. These date from the 1800s, I think, and the mathematics is elementary (i.e. high-school).
 - iii. The insolubility of the first two Delian problems is a good story, but involves most of Math 335. The third problem is too, but it uses some rather involved and delicate theory of infinite series, as in our analysis courses.
- 5. *Parallel postulate*. Because of the general interest in matters related to the parallel postulate, I'll sketch the proof of its equivalence with *Playfair's version*, the form of the postulate most commonly used today.
 - a. Euclid's postulate says that if two lines g and h intersect a line k at points A and B , and E and F are points of g and h on the same side of k , and $m\angle ABF + m\angle BAE < \text{two right angles}$, then g and h intersect on the that same side of k .
 - b. Clearly, this statement is more complicated than Euclid's other postulates. Moreover, its diagram can't be so readily contained within a bounding circle. Euclid avoided using it as long as possible.
 - c. Thus mathematicians asked
 - i. whether it could be proved from the others as a proposition, and if not,
 - ii. whether there might be a simpler but equivalent statement to use as an axiom instead.
 - d. Playfair's proposition says that given a point P and a line g there's at most one line through P and parallel (or equal) to g . In fact, it's equivalent to Euclid's postulate. Playfair was a noted geoscientist. In the book Playfair [1795] 1860, a version of Euclid's *Elements*, he used this proposition in place of the parallel postulate. It has the advantage of not involving angle measure.
 - e. In the late 1800s mathematicians finally established that no version of Euclid's postulate can be derived from the other postulates: some version must be assumed.
- 6. *Existence of a parallel*
 - a. Given a point P and a line g is there at *least* one line h through P and parallel (or equal) to g ? This can be proved without using the parallel postulate, essentially by letting a perpendicular fall from P to a point Q on g , then erecting a perpendicular h to PQ at P . These two operations require justification, as follows.
 - b. *Vertical angle theorem* (Euclid I.15). That vertical angles are equal follows from the observation that each is supplementary to an angle adjacent to both, that the sum of two adjacent angles is a straight angle, which is the sum of two right angles, and any two right angles are equal by a postulate.
 - c. *Midpoint theorem* (Euclid I.10). Construct a midpoint for a given segment AB as follows.
 - i. By proposition 1, construct equilateral triangles $\triangle ABC$, $\triangle ABD$ on both sides of AB . (Euclid failed to discuss the distinction of the sides).

- ii. The *pons asinorum* yields $\angle ACD \cong \angle ADC$, $\angle BCD \cong \angle BDC$, hence $\angle C \cong \angle D$ by addition.
- iii. By SAS, $\triangle ABC \cong \triangle ABD$, and therefore $\angle ABC \cong \angle ABD$, $\angle BAC \cong \angle BAD$.
- iv. By addition, $\angle CAD \cong \angle CBD$.
- v. By SAS, $\triangle CAD \cong \triangle CBD$, and therefore $\angle ACD \cong \angle ADC$.
- vi. CD meets AB at a point M , because C, D lie on different sides of AB (which Euclid didn't mention).
- vii. By SAS, $\triangle CAM \cong \triangle CBM$, and therefore $AM = BM$, as required.
- d. *Exterior angle inequality* (Euclid I.16). Extend edge AB of $\triangle ABC$ past B to D . Then $\angle CBD > \angle C$. Here is a proof.
 - i. Find M , the midpoint of BC , and extend AM past M to a point P such that $AM = MP$.
 - ii. By the vertical angle theorem and SAS, $\triangle AMC \cong \triangle BMP$, and therefore $\angle C \cong \angle MBP < \angle CBD$, as required.