

1. *Struik, chapter 2*
 - a. Agriculture along the great rivers led to development of social organization, cities, urban aristocracy, specialized craftsmen and functionaries, despotism, organized religion.
 - b. The more complex social structures would rise, decline, and be replaced by others, but the underlying village structure remained fairly constant.
 - c. The underlying constancy evidently led to much the same basic mathematics being developed in very different societies and locations. It originated as a practical science to meet the needs of organized society.
 - d. Since religious and administrative functions often overlapped, priests were often the carriers of scientific knowledge.
 - e. *But*: “A science cultivated for centuries by a special craft, whose task it is not only to apply it but also to instruct its secrets, develops tendencies toward abstraction.”
 - f. *But*: Nowhere in this mathematics do we find any semblance of a proof.
2. *Struik on Egypt*
 - a. The Egyptians had a system of numerals rather like Roman numerals.
 - b. Those are fairly easy to add and to double.
 - c. To multiply they would repeatedly double and then add as required. This involves an implicit understanding of the binary numeration system and the distributive law.
 - d. They considered natural numbers plus reciprocals of distinct positive integers.
 - e. They constructed elaborate tables of doubles of reciprocals.
 - f. This allowed them to add such numbers, hence to multiply them by natural numbers.
 - g. They also figured out how to multiply by a reciprocal, and thus how to divide by numbers of this sort.
3. *Sidelight: Every positive rational number r is in fact an integer or the sum of a natural number and some reciprocals of distinct positive integers.*
 - a. The proof is in Burton 2007 (§2.2, 45), where it is attributed to Fibonacci, a Renaissance Italian, about whom we’ll read soon. It amounts to an algorithm, as follows:
 - i. If r is an integer, there is nothing to do.
 - ii. Otherwise, $r = s + a/b$, where s, a, b are natural numbers and $b > a > 0$.
 - iii. Find the positive integer n such that $(n - 1)a < b \leq na$. If $n = 1$ then b would be $\leq a$, but it’s not; thus $n > 1$ and $nb > b$. If $b = na$, then $a/b = 1/n$, and there is nothing more to do. Thus it can be assumed that $b < na$, from which follows $0 < na - b$, and thus $1 \leq na - b < na$. Moreover, from $(n - 1)a < b$ follows $na - b < a$.

- iv. Therefore, $(na - b)/(nb)$ is a positive rational number with numerator smaller than a and denominator larger than b . Moreover,

$$\frac{a}{b} = \frac{1}{n} + \frac{na - b}{nb}.$$
 - v. Now repeat the preceding steps with the right-hand fraction in place of a/b , and continue repeating if possible.
 - vi. That process cannot go on forever, because with each repetition the numerator of the fraction being decomposed decreases. Moreover, the denominators increase, so they are all distinct.
 - b. That kind of argument is an application of the *principle of infinite descent*: a decreasing sequence of natural numbers can't go on forever.
 - c. This principle is closely related to the principle of *mathematical induction*, which was studied intensely and publicized widely by Giuseppe Peano in 1889 (and by some others, whom we'll discuss later).
 - d. Here's how Fibonacci's algorithm would write $34/25$ as such a sum.
 - i. $34/25 = 1 + 9/25$.
 - ii. So we only need consider $9/25 = a/b$.
 - iii. $2a = 18 < b \leq 27 = 3a$, so $n = 3$.
 - iv. $na - b = 27 - 25 = 2$.
 - v. $9/25 = 1/n + (na - b)/(nb) = 1/3 + 2/75$.
 - vi. Now we only need consider $2/75 = a/b$. These new a, b are the previous values of $na - b$ and nb .
 - vii. $37a = 74 < b \leq 76 = 38a$, so corresponding to the new a, b is the new $n = 38$.
 - viii. $na - b = 76 - 75 = 1$.
 - ix. $2/75 = 1/n + (na - b)/(nb) = 1/38 + 1/(38 \cdot 75) = 1/38 + 1/2850$.
 - x. $34/25 = 1 + 1/3 + 1/38 + 1/2850$.
 - e. The Egyptians had such an algorithm, evidently different, with no proof that it always worked.
4. *Struik on Egypt, continued*
- a. Struik said that the Egyptians regarded $(d - d/9)^2 = (64/81)d^2$ as a formula for (let's say *approximating*) the area of a circle with diameter d . We know that $\frac{1}{4}\pi d^2 = \pi(d/2)^2 =$ this area. Thus, the Egyptian formula amounts to saying that $\frac{1}{4}\pi \approx 64/81$, i.e. $\pi \approx 256/81 \approx 3.16$: pretty close.
 - b. He also mentioned that they knew how to compute the volume of a frustum of a pyramid. In the previous lecture I derive the formula he displayed for that. But my derivation required hard algebra and easy calculus, and justifying the calculus would require much more algebraic reasoning. *The Egyptians had no algebra and no calculus.*
5. *Struik on Mesopotamia*
- a. Struik regarded Mesopotamian mathematics as much more advanced than Egyptian. He's vague about dates: his Egyptian dates are older, but overlap the Mesopotamian.

- b. Mesopotamia is centrally located between Greece and India, and certainly served as a source of mathematics in those locations, and a transmitter of mathematics between them. I think it's generally agreed that the Mesopotamian predates the Greek, but there is dispute about which mathematics may have originated first in Mesopotamia and which in India.
- c. The Mesopotamians had a place-value system of numerals, more like ours. That greatly facilitates computation.
- d. The Babylonian period in Mesopotamia evidently began around 1750 B.C.E.
- e. The Babylonians could solve all problems that we would normally solve by setting up and manipulating a linear or quadratic equation. They were also able to solve some special cubic and quartic problems, and they knew the Pythagorean theorem.
- i. Struik pointed out that they had a way of approximating square roots that is roughly equivalent to what we call Newton's method. Suppose you have a coarse approximation $a \approx \sqrt{A}$ (possibly attained by noticing that $a^2 \approx A$). Let $h = A - a^2$, which should be small relative to a , then notice that also

$$\left(a + \frac{h}{2a}\right)^2 = a^2 + h + \frac{h^2}{4a^2} \approx a^2 + h = A.$$

If $a' = a + h/(2a)$, then $a' \approx \sqrt{A}$, too. Is it a closer approximation than a ? Further algebraic manipulation yields $a' = \frac{1}{2}(a + A/a)$ —this is in fact the familiar *Newton's method* for computing square roots, which is discussed in most courses on differential calculus. Moreover,

$$\frac{a' - \sqrt{A}}{(a - \sqrt{A})^2} = \frac{\frac{1}{2}\left(a + \frac{A}{a}\right) - \sqrt{A}}{(a - \sqrt{A})^2} = \frac{1}{2a}.$$

If $a > \sqrt{A}$, then this ratio is less than the constant $c = 1/(2\sqrt{A})$ and the numerator of the middle fraction is $(a - \sqrt{A})^2/(2a) > 0$. Thus, in this case, the numerator of the left-hand fraction is positive, hence $a' > \sqrt{A}$; you can repeat this process with the new approximation a' and get a sequence of approximations all larger than $\sqrt{A}$, and the error in each approximation will be less than c times the *square* of the error in the previous one. Even if the initial approximation is quite coarse, the succeeding ones will converge extremely rapidly to $\sqrt{A}$. (This type of convergence, with the square in the error analysis, is called *quadratic*; you would study such processes in detail in numerical analysis courses.) Finally, if the initial approximation were $< \sqrt{A}$, then it is easy to see that the next one will be $> \sqrt{A}$, and convergence will thus also be quadratic.

- f. Struik also mentioned that the Babylonians knew the equivalent of an interpolation formula for finding x such that $1.2^x = 2$. We would do that by noticing $1.2^3 < 2 < 1.2^4$, so that $3 < x < 4$. Then we would find the equation of the

line through the points $(3, 1.2^3)$ and $(4, 1.2^4)$, regard the line as approximating the graph of 2^x , and as our approximation to the desired x take the abscissa of the point on that line with ordinate 2. As a high-school student I had to become expert at that technique.

6. Struik paid some attention to the mathematics of ancient *India and China*, but I will not add to that here. I am simply not competent to do so. The history of Indian mathematics has been an active field for many decades. Interest in that of East Asia is increasing fast. Each general meeting on the history of mathematics features some talks on that area now.
7. Struik emphasized again that nowhere in any of the mathematics of this chapter is there any appearance of mathematical proof.
8. Some of Struik's terms:
 - a. sum of *geometrical progression* (From $s = 1 + a + \dots + a^n$ deduce $as - s = a^{n+1} - 1$ and derive the formula for s .)
 - b. *parallelepiped*
 - c. *Sothic cycle* ≈ 1460 years
 - d. *sexagesimal* = having to do with 60
 - e. *cuneiform* = wedge-shaped (Peano's home city of *Cuneo* lies in the *wedge* of land at the confluence of two rivers!)
 - f. *ephemerides* = astronomical tables
 - g. *Alexandria* = a major center of Greek culture, in Egypt
9. *Assignment*
 - a. Read Struik [1949] 1987, chapter 3.
 - b. Read J. T. Smith 2000, section 2.3. It's online: [click here](#).